Model Standards in Mathematics for Beginning Teacher Licensing & Development: A Resource for State Dialogue

Developed by
Interstate New Teacher Assessment and Support Consortium
Mathematics Sub-Committee

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Preface

More is expected of our teaching force now than ever before. Successful efforts to restructure schools for the demands of a knowledge-based economy depend critically on the nation's teachers. To better prepare students to live and work in the 21st century, teachers are expected to support the needs of all learners, not just to "cover" the curriculum. Schools are expected to ensure that all students learn and perform at high levels, not just to "offer" a well-rounded education. To meet new expectations for their roles and for the performance of their students, teachers will need a deeper understanding of subject matter than many pre-service programs now provide. Teachers will need a deeper understanding of how knowledge is developed and can be nurtured through well chosen pedagogical strategies. Teachers will need a deeper understanding of how individual differences affect the course of learning and how teaching strategies can be adapted to build on the strengths that individual learners present.

New expectations and the new knowledge base for teaching dictate changes in how teachers are educated and how their continued development is sustained throughout their careers. New expectations also dictate the creation of consistent policies for licensing and certifying teachers and for regulating and accrediting teacher preparation institutions. The Interstate New Teacher Assessment and Support Consortium (INTASC) has focused on standards-based licensing as a lever to promote needed changes throughout the system.

INTASC, now sponsored by the Council of Chief State School Officers, was established in 1987 to enhance collaboration among states interested in reforming teacher preparation and teacher assessment for licensing. Its primary constituency is state education agencies responsible for teacher licensing and professional development, including both state boards of education and professional standards boards. Presently 36 states are actively involved in INTASC work. INTASC also includes representatives of several key national educational organizations such as the National Board for Professional Teaching Standards (NBPTS), the American Association of Colleges of Teacher Education (AACTE), National Council for the Accreditation of Teacher Education (NCATE), National Association of State Boards of Education (NASBE), the National Association of State Directors of Teacher Education and Certification (NASDTEC), the National Education Association (NEA), and the American Federation of Teachers (AFT).

In 1992, INTASC proposed a set of rigorous standards for beginning teachers that could guide coherent policy development in all areas that influence the quality of the teaching force. The model core standards were offered to stimulate discussion about the knowledge, dispositions and performances that are essential for all beginning teachers, regardless of the subject being taught. Release of the core standards marked INTASC's first major milestone in promoting a systemic, performance-based approach to education, licensing, and supporting the continued professional growth of teachers.

The INTASC core standards have been well received and are now broadly recognized as providing a useful framework for reforming many aspects of teacher education, licensing, and professional development. In the last few years, they have been used as the basis for revising licensing and program approval processes in Arkansas, Connecticut, Maine, Minnesota, Ohio and Texas. Other states such as Delaware, Kentucky, Indiana, Pennsylvania, Vermont, and Wisconsin are now engaged in a similar process. In addition, NCATE now uses INTASC core standards as criteria to determine whether a program should be accredited.

Building on this overwhelmingly positive response to the core standards, INTASC has turned its attention to translating the model core standards into discipline-specific standards for teaching in each of the major K-12 disciplines. In addition to these licensing standards in mathematics, INTASC plans standards development efforts in English/language arts, science, music and the arts, history/social studies, elementary teaching and special education. The timetable for work in each area is being determined by the states' interest in, and need for, standards in a particular subject area and by the availability of and consensus on K-12 content standards that can serve as the foundation for discipline-specific INTASC standards.

INTASC's next major milestone is to develop assessments linked to the discipline-specific standards for teaching. This process is underway with ten INTASC states which have made a three-year commitment to collaborate on developing prototype performance assessments in mathematics and English/language arts. The model standards and the assessment prototypes will provide a road map for all INTASC states to reform both the professional preparation of teachers and the state's decision making process to award a teaching license

Introduction

Mathematics Education for the 21st Century

The National Education Goals, established in Goals 2000: Educate America Act of 1994, set the following challenge for the nation:

> Goal 5: By the year 2000, U.S. students will be first in the world in mathematics and science achievement.

Goal 5 rests on the growing national consensus that a mathematically literate citizenry is essential for maintaining our democracy and ensuring a competitive position in an international economy. This has been a common message in the many reports produced by panels and commissions charged with evaluating the status of education and the economy in preparation for the 21st century.

The National Research Council looked to the future to justify the critical need for a change in mathematics education. *Everybody Counts* (1989) began with the following observation:

> *"Not only in mathematics but also in every other school subject, educators are faced with significant demographic changes and rising expectations for preparing the kind of work force the country will need in the future. Information-age technology will continue to grow in importance. Pressed by rising international competition American industry will demand improved quality and increased productivity. The world of work in the twenty-first century will be less manual but more mental; less mechanical but more electronic; less routine but more verbal; and less static but more varied."*
>
> *"Communication has created a world economy in which working smarter is more important than just working harder. Jobs that contribute to this world economy require workers who are mentally fit - workers who are prepared to absorb new ideas, to adapt to change, to cope with ambiguity, to perceive patterns, and to solve unconventional problems. It is these needs, not just the need for calculation (which is now done mostly by machines), that make mathematics a prerequisite to so many jobs. More than ever before, Americans need to think for a living; more than ever before, they need to think mathematically."* (pp 1-2)

The Secretary's Commission on Achieving Necessary Skills (SCANS) Report *What Work Requires of Schools* (1991) from the Department of Labor also pushed for a stronger link between knowledge and work force skills. It emphasized the pragmatic applications of theoretical knowledge and the productive use of resources, interpersonal skills, information, systems, and technology as critical preparation for entry into the work force.

It is clear that real world applications provide a primary reason for learning mathematics. Being educated in mathematics is essential to leading a productive life. Being educated in mathematics means having good number sense, being able to make reasonable estimates and to recognize when a calculator's computation is off by a factor of ten. It is developing spatial sense that helps to visualize how shapes can be taken apart and reassembled, being able to estimate, for example, how much paint is needed for a house without measuring every alcove and every dormer. Being educated in mathematics means understanding the use of data to evaluate information, to determine whether or not you are likely to benefit from a new pain-relief medication based on information from clinical trials. Being educated in mathematics means having an understanding of functions that allows you to compare one-year adjustable rates with fixed-rate mortgages and determine the conditions under which each is a better borrowing practice. These examples represent the real-world situations that command the application of mathematics.

However, the study of mathematics must extend beyond the ways in which we quantify the current world. Mathematics is also the basis for exploring new problems and meeting the challenges of the future, in some instances through abstract or theoretical models. The continued development of mathematics will provide the basis for seeking new solutions to previously unimaginable problems that will be encountered as a result of technological advances in the next century.

Reaching the goal of a mathematically literate citizenry, educated for the demands of the next century, will require fundamental reform of three critical aspects of mathematics education.

- the curriculum -- what is taught in mathematics classrooms;

- the mathematical knowledge of teachers -- mathematical understandings teachers bring to the classroom; and

- the process of instruction -- how mathematics is taught and assessed.

Mathematics teachers have already begun a critical examination of curriculum and instruction. Building on *Everybody Counts*, the National Council of Teachers of Mathematics (NCTM) provided a vision for curriculum in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and for mathematics instruction in *Professional Standards for Teaching Mathematics* (NCTM, 1991). The Mathematics Association of America (MAA) addressed the preparation of mathematics teachers in *A Call for Change*. Mathematics educators also have begun articulating principles for designing assessments that support new directions within the field. The Mathematical Sciences Education Board's (MSEB) *Measuring Up: Prototypes for Mathematics Assessment* and *Measuring What Counts: A Conceptual Guide for Mathematics Assessment* (1993) and the NCTM's *Assessment Standards for School Mathematics* (Working Draft, 1993) provide new insights into assessments in mathematics.

Each of these reports has been instrumental in defining how mathematics and the ways in which it is taught must change to promote mathematics literacy for all students. The INTASC standards for mathematics teaching build on the common vision articulated in these documents.

Framework for the Mathematics Standards: INTASC Core Standards

The standards for mathematics teaching build equally on INTASC model standards, which specify a common core of teaching knowledge and skills to be acquired by all new teachers. This "common core," like the first tier of assessment for licensing in virtually all other professions, is intended to outline the common principles and foundations of practice that cut across specialty areas. For the teaching profession, this core is comprised of several kinds of knowledge, most particularly, of student learning and development, curriculum and teaching, contexts and purposes -- the set of professional understandings, abilities and commitments that all teachers share.

The core standards were developed to embody three essential characteristics. First, they are performance-based; that is, they describe what teachers should know and be able to do, rather than listing courses that prospective teachers must complete in order to be awarded a license. Second, they reflect the kind of teaching required to help students develop the knowledge and skills toward which learner-centered educational reform efforts aim. Third, the core standards are explicitly linked to standards being created by the National Board for Professional Teaching Standards to recognize accomplished teaching.

INTASC undertook standards development by creating a task force to consider how licensing standards for entry into teaching would have to change to be compatible with the work of the National Board. "Board-compatibility" was considered essential to creating coherence in the system of career development for teachers, from initial licensing through certification of accomplished teaching. The task force began its work by examining five propositions that guide the National Board's work:

- Teachers are committed to students and their learning.
- Teachers know the subjects they teach and how to teach those subjects to diverse learners.
- Teachers are responsible for managing and monitoring student learning.
- Teachers think systematically about their practice and learn from experience.
- Teachers are members of learning communities.

The task force arrived at the 10 principles that comprise the model core standards by asking what beginning teachers ought to be prepared to know and to do in order to develop these capacities over time. Each principle was then described further in terms of underlying knowledge, dispositions, and performance skills expected of all new teachers, regardless of the subjects, grade levels, or students taught. These served as the foundation for building the mathematics standards.

Linking the INTASC core standards to the National Board's propositions had two immediate consequences. First, it raised concerns among some that the standards might be unrealistically high for beginning teachers. While the task force acknowledges that the standards are quite rigorous, members viewed them as appropriately so. They argued that for students to meet new content and performance expectations, more would have to be required of all teachers, including those new to the profession. Second, was the question of how to distinguish the practice of a competent beginning teacher from the advanced performance expected of a Board-certified teacher, given the close correspondence between the two sets of standards. On this question, the task force concluded:

> *"... [T]he appropriate distinctions between beginning and advanced practice are in the degree of sophistication teachers exhibit in the application of knowledge rather than in the kind of knowledge needed. Advanced practitioners will have developed their abilities to deal simultaneously with more of the complex facets of the teaching context, with greater flexibility and adaptability, and a more highly-developed capacity to integrate their understandings and performances on behalf of students' individual needs... [Beginning] teachers must have, at the least, an awareness of the kinds of knowledge and understandings needed ... to develop these skills,*

> *must have some capacity to address the many facets of curriculum, classroom, and student life, and must have the dispositions and commitments that pledge them to professional development and responsibility*.[1]

As a practical matter, we expect the differentiation between novice and accomplished levels of performance to emerge as prototype performance assessments are developed to mirror teaching standards for specific content areas.

Overview of the Standards for Teaching Mathematics

The core standards were translated into principles for mathematics teaching by a mathematics subcommittee working under the direction of the Core Standards Task Force. The majority of the participants were currently practicing teachers, most of whom had been involved in the standards work of the NCTM or the National Board. Work on the mathematics standards began by having participants identify the disciplinary and pedagogical concepts deemed essential to competent mathematics teaching by beginning teachers. The subcommittee drew heavily on resource documents produced by the NCTM, the MSEB, the MAA and the National Board, continuing to examine each proposition in relation to the INTASC core standards. What became clear through this process was that the visions set out in these documents not only create new expectations for teachers, but *require a reexamination of who is considered a teacher of mathematics*.

Teachers of Mathematics

The INTASC vision of mathematics instruction provides for the study of challenging mathematics at all grade levels and regards all students as successful learners. This new vision requires a new and more inclusive definition of the term "teacher of mathematics". Included in this designation are all teachers who assume responsibility for the mathematics development of all children, whether they are teaching mathematical foundations in the primary grades, mathematics for special needs students, transition mathematics in the middle school, or advanced mathematics to high school students. All are considered *teachers of mathematics* who share a common responsibility for presenting mathematics as a coherent discipline that builds mathematical understandings across the K-12 curriculum. All will be held to the new expectations being established by the profession.

Teachers who teach mathematics at any grade level will be expected to understand the importance of what they are teaching as a foundation for their students' future learning of mathematics. They will be expected to provide challenging mathematics to students at very different levels of mathematical understanding and to adequately address the full range of student questions that arise from learning diversity. Teachers of mathematics at every grade level will be expected to teach more mathematics and more challenging mathematics to their students, and to be knowledgeable about the curricula that precede and follow the mathematics they teach. They also will be expected to understand that more advanced concepts of mathematics can be taught to students at younger ages and to use developmentally appropriate strategies in helping students explore these concepts at earlier points in the

[1] An expanded discussion of this and other policy decisions that undergird the core standards and the standards for mathematics teaching is presented in the INTASC publication *Model Standards for Beginning Teacher Licensing and Development: A Resource for State Dialogue*.

curriculum.

This is the view of teachers that guided development of the mathematics standards. Following the INTASC core standards, Mathematics Principle 1 describes the content or subject matter base and Principles 2-10 describe the pedagogical base that enables such teaching. An overview of the standards follows, with separate descriptions provided for these two aspects of teaching knowledge and ability.

Knowing Mathematics

Mathematics Principle #1 links the NCTM K-12 curriculum standards to the first INTASC core principle. It requires that teachers who teach mathematics at all grade levels and in all teaching situations have a deep understanding of the critical mathematical ideas, processes and perspectives needed to help all students develop mathematical power.

Mathematical Ideas

Teaching mathematics as a coherent framework across grades K-12 requires that all teachers of mathematics share a common, core knowledge base of mathematical concepts and procedures. That core knowledge base can be organized into the following strands: number systems and number theory; geometry and measurement; statistics and probability; functions, algebra, and concepts of calculus; and discrete mathematics. Teachers of mathematics also develop an understanding of mathematical structures which govern these mathematical strands. Through an understanding of these mathematical concepts, procedures and the structures of mathematics, teachers help students reach new understandings in the process of developing mathematical power.

Teachers know the full mathematics curriculum so that they can build upon the mathematical understandings of their students and can construct a foundation for the mathematics that will follow. At all grade levels, teachers see how the mathematics they are teaching fits into the "big" picture of mathematics, and they understand how major mathematical ideas (e.g., patterns, functions) are developed over the course of a student's progression through the mathematics curriculum.

Mathematical Processes

Teachers perceive mathematics as a dynamic process of identifying, exploring, and solving problems and of developing new understandings to build a knowledge base in mathematics. They understand that problem solving, communication, reasoning, and making connections are the linkages between knowing mathematics and doing mathematics.

Mathematical Perspectives

Teachers understand the historical roots of mathematics. The abstractions of mathematics often had a basis in much simpler problems. Much of mathematics arose from the needs of various cultures to quantify, to communicate, and to solve what initially seemed to be very concrete problems. Teachers know the history of the development of mathematics and remain cognizant of the ever-changing nature of the discipline.

Teachers understand that their view of mathematics has been shaped by the society in which they live. Mathematical world view is dependent on our cultural experiences. It is linguistically, culturally, and environmentally determined. Teachers appreciate that each student may have a basic mathematical world view that is substantially different from others based on their family's linguistic or cultural background, and help all students build on their *own* innate mathematical world view.

Teachers help students see connections as they build from arithmetic understandings to algebraic understandings; as they take simple patterns and rules for these patterns and write equations to represent the functions; as they begin with simple data gathering, representation, and analysis and start to use data in simulations to model real-world phenomena and make predictions and policy recommendations; and as they take their spatial knowledge and ability to measure objects around them and extend these abilities to describe and measure objects which cannot be measured in a concrete way.

Teachers know the ways in which the study of mathematics is changing. They incorporate these changes into their learning and teaching of mathematics as evidence of their efforts to stay current and prepare for the future in mathematics. Technology is one example of a current influence that is constantly changing the ways we perceive mathematics. Technology allows new and different approaches to problem solving. It redefines content and actually creates new mathematics. Technology can facilitate conceptual understanding in mathematics and create new procedures and perspectives for solving mathematical problems. An understanding of mathematics in a technological age is essential to understanding the present and the future of mathematics.

In summary, as the traditional K-12 mathematics curricula are changing, the content knowledge base for teachers must change accordingly. Mathematics Principle 1 aims to assure that every new teacher is grounded solidly enough in the discipline to help students learn more and more powerful mathematics.

Teaching Mathematics: The Pedagogical Understandings

Mathematics Principles 2-10 represent an integration of the INTASC core principles with the NCTM's *Professional Standards for Teaching Mathematics*. They describe the pedagogical understandings and dispositions through which teachers of mathematics translate subject matter knowledge into effective practice in inquiry-based, learner-centered classrooms. In such classrooms exploration replaces rote memorization. Students are actively engaged in the learning and "doing" of mathematics. Through this active exploration of subject matter, conceptual and procedural understanding develop in an integrated way and mathematical ideas and their applications are connected. Conjecturing, inventing, and problem solving are valued. In learner-centered classrooms students "talk mathematics" as well "do mathematics". In these conversations, reasoning, logic, and mathematical evidence provide the basis for discussion.

Teachers who can orchestrate active, inquiry-based instruction have firm foundations in student learning. They know how understanding and competence develop, and how individual student interests, backgrounds and experiences shape learning. They also have specific knowledge of how mathematical ideas and mathematical competence develop, and they know how to nurture this development. They know how to present ideas and organize learning activities in ways that promote development of deep understanding -- which mathematical representations to use, which resources and materials to draw on, and which individual and group strategies to employ.

Teachers of mathematics call upon knowledge of varied kinds and from varied sources to make decisions about what to teach, how to teach, and how to assess mathematics learning. As they plan for instruction and assessment, they bring all of their knowledge into play: knowledge of student learning, learning diversity, mathematical development, instructional pedagogy, and the context of their schools and their communities.

The pedagogical knowledge base also includes important professional dispositions. Teachers of mathematics take an active role in their own professional development. They constantly reflect on their practice. They work collegially with other teachers, parents, and agencies within the community, both to continue growing as professionals and to support the mathematical development of their students.

Summary

Achieving Goal 5 of the National Education Goals -- making U.S. students first in the world in mathematics and science -- rests on meeting one of its most critical objectives: "The number of teachers with a substantive background in mathematics and science will increase by 50 per cent." To satisfy this objective, we must ensure that teachers of mathematics embrace the vision of mathematics education demanded by the 21st century. This vision is predicated on the belief that ALL students can learn powerful mathematics. Teachers who share this vision will help ALL students develop their mathematical understandings to the fullest. The INTASC standards attempt to capture what such new teachers know and do as they create experiences through which learners attain high-level mathematical literacy.

Model Standards for Beginning Teachers of Mathematics

The standards for mathematics teaching are presented in two sections to reflect the distinction between content and pedagogical knowledge. *Knowing Mathematics* presents the content necessary for beginning teachers to teach mathematics at any grade level, mirroring INTASC Core Principle #1. One set of common standards is presented for *all* of these beginning teachers. Additional standards for content knowledge are presented to distinguish the more advanced topics and understandings necessary to teach mathematics at the middle and high school levels.[2]

The *Teaching Mathematics* section describes how pedagogical knowledge, reflected by INTASC core principles 2-10, is translated into teaching decisions and actions. The *Teaching* section is further divided into two main segments. First, each of the nine *Teaching* principles is elaborated briefly. Next, four vignettes are presented to illustrate the interaction of the principles as they play out in teaching particular mathematics content.

Throughout both sections, the standards are expressed in descriptive rather than prescriptive language, to reflect our conviction that the profession cannot be strengthened by telling teachers what they should or must do. We believe, instead, that the profession can be made stronger only by portraying in clear and specific terms what it means to be a well prepared teacher of mathematics.

Knowing Mathematics: Principle 1

Teachers who can help all students reach new and more rigorous standards for mathematics achievement have a deep understanding of the discipline they teach. They understand the important concepts and ideas within their discipline and how the content they teach connects with both more rudimentary and more advanced concepts in the school mathematics curriculum. They have a working knowledge of mathematics that connects procedural with conceptual understanding. They understand the role of mathematics in varied social, historical and technological contexts. INTASC Mathematics Principle 1 identifies the critical mathematical ideas, processes and perspectives upon which such understandings are built.

Principle #1: ***Teachers responsible for mathematics instruction at any level understand the key concepts and procedures of mathematics and have a broad understanding of the K-12 mathematics curriculum. They approach mathematics and the learning of mathematics as more than procedural knowledge. They understand the structures within the discipline, the past and the future of mathematics, and the interaction between technology and the discipline.***

[2] Any delineation of standards for teachers raises a genuine concern about how beginning teachers can acquire the needed knowledge and teaching skills. The dilemma is particularly apparent with regard to preparation to teach in the elementary grades. The INTASC Standards Drafting Committee recognizes the difficulties confronting teacher educators who must prepare ALL teachers to meet rigorous standards in all of the subject areas. As INTASC develops specific standards in other subject areas, the issue of preparation of elementary teachers in all areas will be addressed.

Principle #1 cont.

The teacher of mathematics understands mathematical ideas from the following areas:

- **Number Systems and Number Theory**
- **Geometry and Measurement**
- **Statistics and Probability**
- **Functions, Algebra, and Concepts of Calculus**
- **Discrete Mathematics**

The teacher of mathematics develops a knowledge of mathematics through the following critical processes:

- **Problem Solving in Mathematics**
- **Communication in Mathematics**
- **Reasoning in Mathematics**
- **Mathematical Connections**

The teacher of mathematics develops the following mathematical perspectives:

- **The History of Mathematics**
- **Mathematical World Views**
- **Mathematical Structures**
- **The Role of Technology and Concrete Models in Mathematics**

Mathematical Ideas

Number Systems and Number Theory

Number systems and ideas from number theory provide a basis for mathematics in everyday life. Reading a daily newspaper, making informed decisions when comparing prices, comparing the percentage of a salary increase to the annual inflation rate, balancing a checkbook, understanding crime rates, following a recipe, allocating time to different activities, and estimating tax on a purchase all require quantitative skills. An understanding of numbers includes an appreciation of how numbers are used to represent attributes of real world objects and quantities. Working with numbers requires understanding of what an operation is and when to use which operation including the ability to make estimates and mental computations. A sound understanding of number, or number sense, is critical in mathematics.

Number sense is difficult to define, but at the heart of the definition is an understanding of numbers, magnitude, the relationships between numbers and the relative effect of operations on numbers. Teachers help students learn to view the results of computations from a number sense perspective first. Teachers encourage students to address the reasonableness of their answers. For example, finding the cost of 2/10ths of an hour of a mechanic's time at $42.00 /hour could be viewed from a number sense perspective. When confronted with a response of $84.00 to the computation $42 x 0.2, students learn to notice that a response of $ 84 must be incorrect, since multiplying a positive number (42) by a number smaller that one (0.2) results in a number smaller than 42, not larger. Knowledge of the relations between decimals and fractions helps them equate 0.2 and 1/5, using estimation they know 40 x 1/5 is 8, so the answer of 84 is off by a magnitude of 10.

Teachers help students develop number sense through the use of numbers to quantify concepts in their world. They use problems that involve money, time, or other applications to illustrate the importance of numbers and number sense. They demonstrate the relationships between decimals and common fractions with common sense examples like: ten dimes is the same as $ 1.00; 1 dime is 1/10th of a dollar, and another way of writing 10¢ is as a decimal, 0.10 dollar. Teachers help students to understand how numbers are composed and are able to decompose and recompose numbers in order to solve problems. They know, for example, that a class of 28 students can be separated into 1, 2, 4, 7, 14 or 28 groups of equal size.

Teachers help students develop a variety of computation procedures. Given a problem that involves the division 176 ÷ 8, they might compute in several ways: mentally (176 is 160 + 16: 160 ÷ 8 = 20 and 16 ÷ 8 = 2, so the answer is 20 + 2, or 22; or 176 is 200 - 24, 200 ÷ 8 = 25 and 24 ÷ 8 = 3, so the answer is 25 - 3, 22); in written format using pencil and paper; using physical models such as manipulatives to visualize the operation; or using a calculator.

```
      2 2
8   1 7 6
    1 6 0
    -----
      1 6
      1 6
    -----
        0
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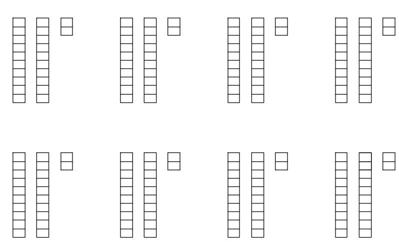


The concepts of number theory (e.g., divisibilty, factors, multiples, prime numbers) provide a basis for exploring number relationships. The use of codes to encrypt or to create security systems for computers depends on number theory. The study of integers and their properties illustrates relationships that can be explored and generalized to other mathematical domains.

Quantitative thought and understanding are critical to fully participate in our society and in the workplace. The underlying structure of mathematics -- Number Systems and Number Theory -- ties mathematics into a coherent field of study, rather than an isolated set of rules, facts, and formulae. Teachers understand and appreciate mathematics as a collection of systems with commonalities across integers and rationals. They take their understandings of the rational number system and extend them to real and complex numbers and beyond. Teachers appreciate how these understandings provide a foundation for all mathematics.

All teachers of mathematics understand number systems and number theory from both abstract and concrete perspectives and are able to identify real world applications. They know the following mathematical concepts and procedures, and the connections among them:

Mathematical Concept:	Examples:
Number sense, including a sense of magnitude, mental mathematics, estimation, place value, and a sense of the reasonableness of results.	*recognizing that the sum of two fractions 5/6 + 7/8 will be close to but less than 2, because each number is almost 1; knowing that 59+44 can not be 913, because both numbers are less than 100; knowing any five digit whole number is greater than any four digit number*
Composition, decomposition, and recomposition of numbers, including place value, primes, factors, multiples, inverses, and the extension of these concepts throughout mathematics.	*recognizing 427 + 303 is equal to 430 + 300; recognizing 0, 3, 6, and 9 as multiples of 3; seeing* x^2+3x+2 *as* $(x+2)(x+1)$ *or 29 as a prime number with only 1 and 29 as factors*
The use of numbers to quantify and describe phenomena such as time, temperature, money, and trends.	*finding the elapsed time from 11:15 A.M. till 1:00 P.M. and describing it as 1¾ hours; reviewing a student's test grades of 68, 72, 84, and 87 and noticing an upward trend*
Number systems, their properties and relations, including whole numbers, integers, rational numbers, real numbers, and complex numbers.	*recognizing that the order in which you add or multiply numbers is irrelevant, but is relevant in subtraction and division; knowing that the operation of division extends the integers to the rational numbers;*

Mathematical Concept:

Examples:

Computational procedures, including common and uncommon algorithms; mental strategies; use of manipulatives and other representations; paper and pencil; and technology; and the role of each.

recognizing you can find 345+139 by adding from left to right, e.g. (340+130) + (5+9) = 470 + 14 = 484

Ratio, proportion, and percent as ways to represent relationships.

recognizing that 12/100 can be written as 0.12, 3/25, or as 12% and knowing there are times when the use of one form is preferable for calculation or communication, e.g. boys:girls, 3:5; 3/5ths of the cake; 6/10 of a km; 60% of the voters

Teachers of upper level mathematics know the following additional mathematical concepts and procedures, and the connections among them:

Mathematical Concept:

Examples:

Geometric and polar representations of complex numbers and the interpretation of complex solutions to equations.

representing complex numbers in the form a+bi or r(cos2+i sin2)

Algebraic and transcendental numbers.

identifying real or complex numbers that are roots of polynomial equations having rational coefficients; and non-algebraic numbers such as e, B, 2^3

Numerical approximation techniques as a basis for numerical integration, fractals, numerical based proofs.

knowing what happens to the sequence

$\sqrt{.5}, \sqrt{\sqrt{.5}}, \sqrt{\sqrt{\sqrt{.5}}}, \ldots;$

Number theory divisibility, properties of prime and composite numbers, the Euclidean algorithm.

Pythagorean triples, three integers in which the square of the largest is equal to the sum of the squares of the other two; understanding the statement of Fermat's Last Theorem, that for n, an integer greater than 2, there are no positive integral values x, y, and z, such that $x^n + y^n = z^n$

Geometry and Measurement

Geometry pervades our daily life. Comparing two products packaged in different shapes, deciding the best way to pack items for shipping, designing a house for maximum living space with minimal lumber costs, cutting a pie into equal servings, determining the best way to fit an oversized package through your car door, charting airline routes, making a design for an art project, and replicating patterns by making symmetric designs in a quilt are all examples of tasks that call for an understanding of geometry. We live in a geometric world.

An understanding of geometry begins with an understanding of shapes in our world, then extends beyond simple description and includes dimensions, position and location, orientation and motion. Critical to an understanding of geometry is a well developed spatial sense. Teachers help students develop spatial sense by working with models to show how changes in shape change other attributes. Eight blocks, each representing a cubic inch could be stacked in a 2x2x2 cube to create a surface area of 24 square inches and a volume of 8 cubic inches, or the same blocks could be spread out to a rectangular solid 4x2x1 to create a surface area of 28 square inches and a volume of 8 cubic inches. Teachers use spatial sense to help students explore geometric concepts by posing questions such as: How else could the eight cubes be combined while maintaining the volume of 8 cubic inches? Can you cut the rectangular solid into other shapes? Can you build two congruent solids? Think about an orange as a model of the earth. Locate three cities on your "earth" and connect them, creating a spherical triangle. What happens to the interior angles of a triangle drawn on the orange peel when that peel is opened up?

Teachers help students use spatial sense to explore spatial thinking and reasoning. They help students see the world through a geometric eye, including exploring motion as a geometric concept. How can the same set of tangrams be used to make a triangle, a square and an octagon? How can you fit a table that is 4 feet wide through a 34 inch doorway? Their understandings include rotations, reflections, and translations and the ways in which art, such as Escher's tessellations, incorporate these ideas. Teachers understand how the transformation of shapes can be the basis for exploring concepts of similarity, congruence and other geometric properties.

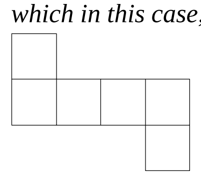
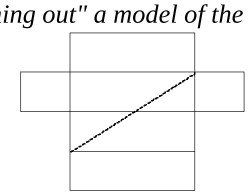
Through geometry, teachers focus on the concepts of logic and reasoning. Geometry provides a systemic way for making assumptions, formulating and testing hypotheses, and justifying arguments in informal and formal ways. Teachers are also able to use more formal geometries, building on the definitions of segments, angles, triangles, spheres, etc., and using the ideas of Descartes to represent geometric concepts on the coordinate plane or in space.

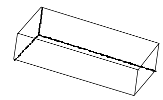
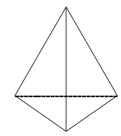
Measurement is the basis by which we quantify our world. Reading a map, taking a pulse, cooking meals, comparative shopping, construction, and many sports all require some form of measurement. Measurement includes identifying attributes to be measured, selecting appropriate units and tools, taking measurements, and communicating ideas based on measurement.

Teachers are able to estimate measurements and take measurements as a way to describe attributes of geometric shapes. Teachers know measurement formulas and measurement procedures for objects that can not be measured directly. Teachers understand that measurement tools are inherently imprecise, and their selection of a measuring tool is dependent on the precision required for a given measurement. Teachers recognize the impact of measurement error. For example, if the measure of one side of a rectangle is off by 3 cm, the measure of its perimeter is off by 6 cm.

Teachers also understand measurement systems, including standard measurement systems like the U.S. Customary Systems, the metric system, and non-standard systems designed to meet specific measurement needs. They understand the properties of the systems, including rules governing combinations of measurements within and across systems. Through dimensional analysis, teachers are able to solve complex measurement problems and their related conversions.

Teachers of mathematics understand geometry and measurement from both abstract and concrete perspectives and are able to identify real world applications. They know the following mathematical concepts and procedures, and the connections among them:

Mathematical Concept:	**Examples:**
Shapes and the ways in which they can be derived and described in terms of dimension, direction, orientation, perspective, and relationships among these properties.	*building shapes, such as points, lines, circles, squares, cubes, pyramids, spheres and using properties to designate equivalent shapes*
Spatial sense and the ways in which shapes can be visualized, combined, subdivided, and changed to illustrate concepts, properties, and relationships.	*taking a piece of string and having students use it as a constant perimeter, while changing shapes (e.g. number of vertices and sides) to create different areas; taking a two dimensional object, like the one below, and visualizing the corresponding three-dimensional object, which in this case, would be a cube* 
Spatial reasoning and the use of geometric models to represent, visualize, and solve problems.	*addressing a problem like "A spider is in a corner of a rectangular room on the ceiling. She spots a fly on the floor in the opposite corner. What is the most direct route she can walk from her spot to the fly? by "flattening out" a model of the room* 

Mathematical Concept:	**Examples:**
Motion and the ways in which rotation, reflection, and translation of shapes can illustrate concepts, properties, and relationships.	*using concepts of turns, flips, and glides within a plane or in space to explore symmetry, patterns, and generalizations; using shape in the process of weaving a Navajo rug; explaining which of the following constructions is more stable:*  
Formal and informal argument, including the processes of: making assumptions; formulating, testing, and reformulating conjectures; justifying arguments based on geometric figures; evaluating the arguments of others.	*offering a convincing explanation of why two shapes are similar; making a conjecture that compares the volume of two objects; generalizing the process for finding the number of diagonals of a regular polygon; justifying claims in formal argument*
Plane, solid, and coordinate geometry systems including relations between coordinate and synthetic geometry, and generalizing geometric principles from a 2-dimensional system to a 3-dimensional system.	*describing the distance between Quito, Ecuador and Macapa, Brazil (two equatorial cities): in terms of a plane (if you could slice through the earth); in terms of solid geometry (remember you need to cross the Andes Mountains; and in terms of a coordinate system (using longitude and latitude)*
Attributes of shapes and objects that can be measured, such as length, area, volume, capacity, size of angles, weight, and mass.	*describing ways in which shapes and objects can be measured, such as a can of soup, in terms of: height; area of the top; surface area; volume*
The structure of systems of measurement, including the development and use of measurement systems and the relationships among different systems.	*selecting an appropriate system and knowing rules for conversions within and across systems; determining approximately how long it will take to travel the 250 km distance from Quebec to Montreal, at 50 miles per hour*
Measuring, estimating and using measurements to describe and compare geometric phenomena.	*selecting an appropriate unit and an appropriate tool to measure; selecting a unit to compare the angle of ascent of two airplanes; selecting a micrometer, not a meter stick, to measure the thickness of wire*

Mathematical Concept:	**Examples:**
Indirect measurement, and its uses, including developing formulas and procedures for determining measures to solve problems.	*finding the area of a large tiled floor by finding the area of one tile and the total number of tiles; finding the height of a flagpole by measuring its shadow and equating the ratio of the flagpole and its shadow length to the ratio of the length of another object and its shadow*

Teachers of upper level mathematics know the following additional mathematical concepts and procedures, and the connections among them:

Mathematical Concept:	**Examples:**
Systems of geometry, including Euclidean, non-Euclidean, coordinate, transformational, and projective geometry.	*systems of geometry that use logical argument to construct theorems based on a set of assumptions or axioms; postulates and axioms in Euclidean Plane Geometry; finite geometries; recognizing the significance of the parallel postulate and its impact on different geometries*
Transformations, coordinates, and vectors and their use in problem solving.	*the interplay between algebra and geometry, such as the representation of a transformation as a function, enables multiple representations of problems*
Three dimensional geometry and its generalization to other dimensions.	*knowing that a sphere in space with its center at the origin is represented by* $x^2 + y^2 + z^2 = r^2$*, being able to look at its intersection with a plane, such as* $x + y + z = k$
Topology, including topological properties and transformations.	*creating a Mobius strip by giving a single twist to a rectangular strip and joining the ends, which yields only one surface; or imagining a Klein bottle, which is three dimensional, but has only one surface and understanding the properties of these surfaces*

Statistics and Probability

An informed citizenry is dependent on actively collecting and interpreting information. Individuals must be able to critically evaluate arguments and judgments presented to them based on the statistical data provided. Data gathering, representation, and analysis influence everyday decisions such as what soap to buy, what movie to see or what book to read, what career to pursue, or which candidate deserves a vote. Data are also integral to informing business decisions and governmental policy decisions based on market research, opinion polls, or other national indicators. Marketing of major products can include distortions that require consumers who can critically evaluate data and statistical claims. Everyone must be able to read the daily newspaper and evaluate information presented as fact.

It is important that all teachers develop an awareness of the concepts and processes of statistics and probability. They recognize issues that students can explore through data collection and issues that allow students to formulate questions and structure an investigation. They help students collect relevant data, represent that data in summary form, analyze the data and based on that analysis, make inferences, predictions, recommendations, and/or a decision. The purpose might be as simple as determining student preferences for time of day to study mathematics or conducting and evaluating teacher's recommendations regarding textbook selection. Statistics and probability highlight the importance of questioning, conjecturing, and searching for relationships when formulating and solving real world problems. Communication skills are reinforced as students discuss and write about their problems and conclusions.

Teachers help students learn to evaluate arguments and claims based on an analysis of data, such as the relationship between study time and grades, the results of a survey of likely voters in a presidential election, or a summary of indicators of consumer confidence in the United States economy. They help students investigate sampling and the concept of randomness and their importance in probability and statistical claims.

The world in which students live is filled with uncertainties. Teachers help students learn to reduce these uncertainties through predictions based on empirical and/or theoretical probabilities. If the number of reported cases of a disease increases, is the increase significant and should you be concerned? If you are one of five candidates for one job and one of ten for another, are your chances greater of getting an offer from the first company or the second? What if you know that the second company tends to hire graduates of your high school and there is only one other finalist from your high school? Probability is one way of addressing risk in daily living.

Teachers help students understand probability as a way of describing chance in simple and compound events. For example, teachers explain how probability based models, based on actual data, influence what individuals pay for auto insurance premiums and can use these data to explain why adolescents pay more than adults. They are able to connect changes in driving habits with periodic changes in rates. Teachers are able to design and implement probability experiments and simulations, such as combinations from rolling number cubes. Teachers help students learn to predict the likelihood of an event and how to use data to make predictions. They help their students compare the results of events with predictions and explain any differences.

Teachers of mathematics understand statistics and probability from both abstract and concrete perspectives and are able to identify real-world applications. They know the following mathematical concepts and procedures and the connections among them:

Mathematical Concept:	Examples:
Data and its power as a way to explore questions and issues in our world.	*exploring issues such as population growth and its relation to population density under various economic and environmental conditions*
Investigation through data, including formulating a problem; devising a plan to collect data; and systematically collecting, recording, and organizing data.	*conducting surveys and recording and organizing responses; tallying the frequency of phenomena like number of sunny days or car colors; or any data collection designed to answer a question*
Data representation to describe data distributions, central tendency, and variance through appropriate use of graphs, tables, and summary statistics.	*displaying and communicating the message inherent in the data collected, including the range of responses and the most common responses, through representations such as bar graphs, tables, stem and leaf plots, box and whisker plots, scatter plots, and other summary descriptions*
Analysis and interpretation of data, including summarizing data, and making or evaluating arguments, predictions, recommendations, or decisions based on an analysis of the data.	*asking students their favorite types of pizza to decide what kinds of topping a class should order for a pizza party; predicting an election based on a survey of a small group; surveying class SAT or ACT scores and summarizing data in terms of "average" responses (mean, median, mode), distributions - (range, standard deviation), and possible relations (equations, curve fitting); being able to critically read and evaluate the arguments and claims of articles in professional journals and newspapers*
Inference, and the role of randomness and sampling in statistical claims about populations.	*understanding the importance of selecting a sample that is representative of the population, for example, knowing that beliefs of a sample of citizens who call in their responses is not necessarily representative of the entire population; recognizing that national data may not be representative of data for a section of the country*

Mathematical Concept:	Examples:
Probability as a way to describe chance or risk in simple and compound events.	*knowing that when a fair coin is tossed the fraction of tosses that are heads approaches ½ as the number of flips increases; knowing that the chance of throwing a six with two number cubes is 5 out of 36; recognizing if a fair coin is tossed 20 times, and each toss comes up a head, the probability of a head on the 21st toss is still 1/2*
Predicting outcomes based on exploration of probability through data collection, experiments, and simulations.	*generating hypothetical probabilities by repeating an event, such as choosing colored marbles out of a bag or generating random numbers, recording the data, identifying a pattern, and using that pattern to predict future outcomes*
Predicting outcomes based on theoretical probabilities, and comparing mathematical expectations with experimental results.	*generating models that replicate patterns or distributions, like disease infection to determine probabilities of infection for future generations*

Teachers of upper level mathematics must also understand the following additional mathematical concepts and procedures, and the connections among them:

Mathematical Concept:	Examples:
Random variable and its application to generate and interpret probability distributions.	*using a random number generator as a basis for probability models*
Descriptive and inferential statistics, including validity and reliability.	*comparing predictions, making connections and analyzing possible causes for the difference; for example knowing that 40% of all teachers hold a master's degree and then finding only 25% of the teachers in one school hold a master's degree, identifying possible reasons for the apparent discrepancy*
Probability theory and its link to inferential statistics.	
Discrete and continuous probability distributions as a basis for making inferences about population.	*understanding the normal distribution and its role in inference; understanding other distributions, such as the binomial distribution and the Poisson distribution*

Observing patterns is at the heart of every discipline. In language arts, certain patterns emerge in writing; in social studies comparing and contrasting ideas requires the identification of similarities and differences; and in mathematics patterns provide the basis for functions, an important unifying idea in mathematics. Predicting wages based on number of hours worked or, total savings based on a set pattern or rate of interest, knowing how building costs will increase as square footage of a house increases, determining when a car will stop given the rate of braking, or estimating long term effects of infectious diseases based on known infection rates all become possible once a pattern is identified and the relationship is described in terms of what changed and how it changed. The concepts of patterns and functions help to define relationships between two variables, a relationship which is usually identified in terms of change.

Mathematicians are constantly identifying sources of change, trying to identify patterns in that change and seeking mathematical ways to describe what occurs. The concept of change permeates mathematics. It is present in patterns that emerge from problem situations and in the relations and functions that evolve to describe patterns. The concepts of limit, continuity, rate of change and summation provide the basis for studying continuous change in calculus.

Teachers help students identify functions and represent them in a variety of ways. For example, Jeremy buys a candy bar and eats ½ of the remaining candy bar each day. The fraction of the candy bar left at the end of each day can be described as relation between days and candy bars. The table below describes the effect of eating ½ of the remaining candy bars for a period of 5 days.

Day	**Candy Bar Remaining**
1	1/2
2	1/4
3	1/8
4	1/16
5	1/32

1

candy bar remaining 1/2

0 1 2 3 4 5

days

The graph above shows another way students might represent the relation which can also be written in the form $y = (½)^x$ for x = 0, 1, 2, ... Teachers should help students see each of these ways of representing a function, the relationship between them, and recognize the value of each representation.

Representing patterns and functions with symbols, such as the equation, $y = (½)^x$, demonstrates the use of algebraic notation. This notation provides the basis for mathematical abstraction and the study of algebra. Symbol sense and the ability to use symbols to communicate effectively and efficiently are essential for all teachers of mathematics. The use of formal mathematical symbolism

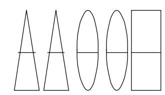
provides the basis for the study of algebra. The symbols of algebra provide a language for performing operations on algebraic expressions and finding solutions to equations, inequalities, and systems of equations. The language provides a means of operating with concepts at an abstract level and taking specific examples to broad generalizations.

Teachers of mathematics understand functions, algebra, and basic concepts underlying calculus from both abstract and concrete perspectives and are able to provide real world applications. They know the following mathematical concepts and procedures, and the connections among them:

Mathematical Concept: | **Examples:**

Patterns, including an ability to recognize, extend, analyze, and describe a wide variety of patterns, realizing that no sequence is uniquely determined by a finite number of terms.

identifying plausible next terms for: 2, 4, 7, 11, ...; or 8, 4, 2, 1, ½, ¼, ...; or $x + 1$, $(x+1)^2$, $(x+1)^3$ *...; or*



Functions and their use to describe relations and to model a variety of real world situations.

identifying step functions for positive integers, such as the cost of postage stamps, 29¢, 58¢, 87¢, for any whole number x, y=.29x; or a linear function, Sam earns $5.50 for every hour of baby sitting, for 1 hour she earns $5.50, for 2 hours $11.00, her pay can be expressed Pay = $5.50 x Number of hours; comparing exponential ($y=2^x$) *and linear* ($y=2x$) *growth*

Representations of situations that involve variable quantities with expressions, equations, and inequalities, including algebraic and geometric relationships.

using symbols or pattern blocks to express 3 pens plus 1 pencil sharpener costs $8.00, 3x+y=8 communicates the abstract relationship; representing the sum of angles 1, 2, and 3 in a triangle by mË 1 + mË2 + mË3 = 180^0*; representing the relationship between pressure, volume and temperature by*

$$P = \frac{kT}{v}$$

using curve fitting to approximate relationships

Mathematical Concept: | **Examples:**

Multiple representation of relations by tables, graphs, words, and symbols, the strengths and limitations of each representation, and converting from one representation to another.

recognizing the relationships among the sequence 2, 4, 8, 16 ; the chart

Days	*Dollars*
1	*2*
2	*4*
3	*8*
4	*16*
	...

the statement "you multiply by two to get the next term"; and the symbols $f(x) = 2^x$

Attributes of polynomial, rational, trigonometric, algebraic, and exponential functions.

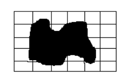
knowing f(x) = x is defined for positive numbers and zero; knowing y = sin x varies periodically from -1 to 1

Operations on expressions and ways to find solutions to equations, systems of equations and inequalities using concrete, informal, and formal methods.

maintaining the balance of an equation, isolating variables, combining like terms, to find solutions to equations like $x + 3 = 7$, $x^2 + 2x = 3$, *or ? x 5 = 30; finding solutions to systems of equations like y = 2x + 1 and y = ½x + 7 or y = 2x + 3 and y = 2x + 4; knowing that some equations and systems of equations may have no solution, a singular solution, or multiple solutions*

Basic concepts underlying calculus, such as rate of change limits and approximations for irregular areas.

approximating area using a square overlay on an irregular figure;



finding the largest volume box that can be made with an 18" by 14" piece of cardboard by cutting squares for each corner of the sheet and folding up the remaining pieces as sides to make an open box

Teachers of upper level mathematics must also understand the following additional mathematical concepts and procedures, and the connections among them:

Mathematical Concept: | **Examples:**

Modeling to solve problems, to understand the behavior of a system or event, and to predict future behavior based on past experiences, to describe a system.

analyzing AIDS data to find a model to predict the spread of the disease; attempting to find a model to predict earthquakes; finding an efficient way to maximize the number of ships that can pass through Suez canal which is a 1-way channel with only two passing bays

Mathematical Concept: | **Examples:**

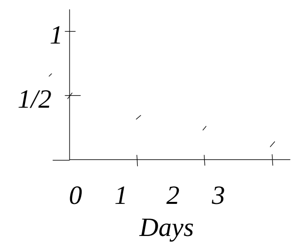
Transformations of algebraic relationships including the effects of the transformation.

transforming data by taking logarithms, e.g., the Richter scale transforms the magnitude of earthquakes to a logarithmic scale; identifying a family of functions such as: $y = x^2$, $y = x^2 - 2$, $y = (x-2)^2$, $y = -x^2$, $y = 2x^2$

Limit, continuity, derivative and integral and their relation to the calculus.

illustrating limits graphically, such as $y=(½)^x$

Absorption of Medication



recognizing the Riemann sum is an integral; graphing discontinuous functions like $y = x/(x^2-1)$; finding the solution to differential equations

Calculus and advanced calculus to model and solve problems involving rates of change, optimization, measurement, and convergence of series.

finding the minimum distance between a point and a curve; finding the minimum surface area for a given volume; finding the maximum volume for a package given the dimensional restrictions from a package delivery service; finding the rate of change in water level in a tank based on in flow and out flow data

Formal study of functions, including multivariate functions, and functions defined with other than Cartesian coordinates.

using DeMoivre's theorem to find the complex roots of real numbers; analyzing functions such as f(x,y)=z and their geometric representations; using parametrics to describe motion

Linear algebra including vectors and vector spaces and their use as a way to organize mathematical relationships that characterize a domain.

recognizing orthogonal vectors and their use to describe relations in three dimensions; using matrices to describe relationships; using algebraic structures such as groups, rings and fields

Discrete Mathematics -- the study of mathematical properties of sets and systems that have a finite number of elements -- is an area of mathematics that has taken on greater importance with the advances in technology. The development of computers as information processors, which are essentially finite, discrete machines, has provided a basis for addressing mathematical problems that previously took too long to compute manually.

Discrete mathematics involves problems such as: determining if an answer exists; systematic counting; determining how many possible solutions exist; and optimization -- finding a best solution to a particular problem. Teachers help students learn to address these everyday problems through discrete mathematical processes. Students should learn how to find the number of combinations possible if you can choose 1 appetizer, 1 entree, and 1 dessert from a list of 3 options in each category or determine how many ways a class of 28 students can be paired off. They should be able to describe all possible ways of getting "12" using three number-cubes as a basis for generating probability models. Students should be able to design a computer programming code that iteratively repeats a process thousands of times, a task that would be too time consuming by hand, to uncover a pattern.

Teachers of mathematics understand discrete processes from both abstract and concrete perspectives and are able to identify real world applications. They know the following mathematical concepts and procedures, and the connections among them:

Mathematical Concept:

Examples:

Counting techniques, including counting arguments to collect and organize information and solve problems.

identifying addition and multiplication principles, such as permutations and combinations, finding how many combinations of 2 scoops of ice cream can you choose from 6 flavors of ice cream; using Venn diagrams to organize information

Representation and analysis of discrete mathematics problems, using sequences, graph theory, arrays, and networks.

using arrays to document T-shirt orders;

size	red	blue	green	total
small	1	2	3	6
med	2	0	0	2
large	0	1	0	1
total	3	3	3	9

using color-coded graphs to identify conflicts in a master schedule

Mathematical Concept:

Iteration, repeating a pattern over and over, **and recursion,** expressing each term as the function of one or more of the previous terms, as processes to generate patterns.

Examples:

seeing the pattern of 1,3,4,7,11, ... where each term is the sum of the previous 2 terms; defining the nth term of S_n *as the sum of the previous two* ($S_n = S_{n-1} + S_{n-2}$)*; using the speed of a computer to repeat this pattern to identify the 50th, 89th, or 1000th term; modeling marketing effects of potential passenger and revenue loss based on each 5% increase in airfares and the resulting loss of 2% of ridership or modeling the spread of infection using iteration*

Teachers of upper level mathematics must also understand the following additional mathematical concepts and procedures, and the connections among them:

Mathematical Concept:

Examples:

Discrete mathematical topics such as symbolic logic, induction, linear programming, and finite graphs.

Matrices as a mathematical system, and matrices and matrix operations as tools to record information and to find solutions of systems of equations.

using arrays of numbers to represent tallies of votes {0, 12, 15, 7} or using matrix operations to find solutions to systems of equations

$$\begin{vmatrix} 1 & 1 & 1 & 6 \\ 2 & 0 & 1 & 5 \\ 1 & 1 & 0 & 3 \end{vmatrix} \div \begin{vmatrix} 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{vmatrix}$$

Developing and analyzing algorithms, including iterative and recursive techniques.

Mathematical Processes

Teachers know more than mathematical concepts and procedures. They are able to reason mathematically, solve problems, and communicate mathematics effectively at different levels of formality. They know the connections among mathematical concepts and procedures as well as their application to the real world. They appreciate the relationship between mathematics and other fields -- the contributions other fields have made to mathematics and the ways in which mathematics has influenced the development of other fields. These four processes -- problem solving, reasoning, communication, and connections -- are essential to understanding mathematics and to developing students' understanding of and appreciation for the value of mathematics.

Problem Solving in Mathematics

> *"Teaching mathematics from a problem solving perspective entails more than solving non-routine, but often isolated problems or typical textbook types of problems. It involves the notion that the very essence of studying mathematics is itself an exercise in exploring, conjecturing, examining and testing - all aspects of problem solving." (NCTM, Professional Standards for Teaching Mathematics p. 95)*

Problem solving is an important reason for studying mathematics. Teachers should experience mathematics as a process of identifying, exploring, and solving problems and developing new understandings to build a knowledge base in mathematics. Tacit knowledge of abstract concepts and algorithmic approaches to solving problems is not enough. Teachers should have experienced struggling to create mathematical understandings. The knowledge thus gained will help them to model problem solving and to create learning experiences for their students that can empower them as problem solvers.

Teachers who teach mathematics at any grade level are able to:

! formulate and pose problems;

! solve problems using different strategies, verify and interpret results, and generalize solutions;

! use problem solving approaches to investigate and understand mathematics; and

! apply mathematical modeling to real-world situations.

Reasoning in Mathematics

> *"Teaching mathematics as an exercise in reasoning should be commonplace in the classroom. Students should have frequent opportunities to engage in mathematical discussions in which reasoning is valued. Students should be able to explain the reasoning process for reaching a given conclusion or to justify why their particular approach to a problem is appropriate. The goal of emphasizing reasoning in the teaching of mathematics is to empower students to reach conclusions and justify statements on their own rather than to rely solely on the authority of a teacher or a textbook." (NCTM, Professional Standards for Teaching Mathematics, p. 96)*

Teachers and their students should understand that struggling with mathematical problems, making false starts, and rejecting hypotheses are part of the process of learning mathematics. Their reasoning, both what they choose to do and what they choose not to do, is an indication of the depth of their understanding of mathematics. Teachers examine patterns, make abstractions and generalizations, and offer convincing arguments. They are prepared to support their argument with examples and to reject other ideas through counter-examples. They are able to make assumptions, develop evidence, and construct proofs. Their ability to reason can be reflected in informal mathematical exploration as well as formal proof.

Teachers who teach mathematics at any level are able to:

- examine patterns, abstract, generalize, and make convincing mathematical arguments;
- frame mathematical questions and conjectures, formulate counter-examples, construct and evaluate arguments; and
- use intuitive, informal exploration and formal proof.

Communication in Mathematics

> *"Communication is the vehicle by which teachers and students can appreciate mathematics as the processes of problem solving and reasoning. But communication is also important in itself, since students must learn to describe phenomena through various written, oral, and visual forms. Mathematics is learned in a social context, one in which discussing ideas is valued classrooms should be characterized by conversations about mathematics..." (NCTM, Professional Standards for Teaching Mathematics, p. 96)*

Mathematical discourse is critical to learning mathematics and communicating about what is being learned. Learning mathematics requires questioning, listening to the ideas of others, clarifying and justifying ideas, and generally communicating thinking orally and in writing. Knowing mathematics and communication in mathematics include knowing the mathematical notation and language that are used to express mathematical ideas. The discourse through which mathematics is learned provides an opportunity to share thinking in mathematics. This dialogue is the basis of learning in any mathematics community.

Teachers who teach mathematics at any level are able to:

- express mathematical ideas orally, in writing, and visually;
- use the power of mathematical language, notation, and symbolism; and
- translate mathematical ideas into mathematical language, notation, and symbols.

Mathematical Connections

> *"The teacher should engage students in a series of tasks that involve interrelationships among mathematical concepts and procedures. The acquisition of mathematical concepts and procedures means little if the content is learned in an isolated way in which connections among various mathematical ideas are neglected. ... Connections should occur frequently enough to influence students' beliefs about the value of mathematics in society and its contributions to other disciplines. Students should have the opportunity to apply the mathematics they have learned to real-world situations that go beyond the usual textbook word problems. Students should see mathematics as something that permeates society and, indeed, their own lives." (NCTM, Professional Standards for Teaching Mathematics, pp. 89-90)*

Teachers who teach mathematics view it as a connected discipline that is not a set of abstract, isolated principles, but is a part of our every day life. They are able to identify the connections within mathematics and to other disciplines and help students begin to identify these connections and the connections between mathematics and their daily lives.

Teachers who teach mathematics at any level are able to:

- demonstrate the interconnectedness of the concepts and procedures of mathematics;
- make connections between mathematics and other disciplines;
- make connections between mathematics and daily living; and
- make connections between equivalent representations of the same concept.

Mathematical Perspectives

Teachers who teach mathematics also develop a number of perspectives on their discipline which help them make mathematics meaningful for their students. They develop an understanding of the history of mathematics; and the interaction between different cultures and mathematics. They can view mathematics through several structural lenses and use these different connections to integrate mathematical concepts and procedures into their students' knowledge bases. Teachers of mathematics adapt to technological changes that can further student's learning in mathematics and incorporate these advances into their instruction.

The History of Mathematics

It is important that teachers have a sense of the history and the future of mathematics. Mathematics has developed as a discipline over time. Teachers who teach mathematics understand its historical roots, in particular the real world basis in its development. They help students to understand that mathematics arose from the need to quantify, to communicate, and to solve what initially seemed to be very concrete problems. The abstractions of mathematics often had a basis in much simpler problems. Likewise, abstractions that are developed as "pure" mathematics often turn out to have applications.

Examples of the historical bases of mathematics help students understand the importance of mathematics in everyday life. Teachers help students understand the history of mathematics by drawing upon examples such as:

- The use of a decimal numeration system has its basis in the fact that most people were born with ten fingers. Early counting systems tended to be based on five digits, quinary; ten digits, decimal; or using fingers and toes, twenty digits, vigesimal. Studies of early American Indian tribes show about one-third used decimal systems, another third used quinary decimal systems, and another ten percent used vigesimal systems.

- The Egyptians noticed a pattern between changes in the stars and the flooding of the Nile. They noticed that shortly after the dogstar rose in the east, before the sun, the Nile would overflow its banks. They tracked the number of days between occurrences and used that number - 365 - as the basis of their year. Over time the seasons seemed to shift and later generations realized that the calendar year was off by 1/4 day. Mathematicians today continue to monitor the estimates for a year and have taken early approximations to even greater precision.

- Algebra was initially developed without the notation that we use today. Mathematicians began the exploration of this area of mathematics with everything written out in words. Diophantus began to use abbreviations in *Arithmetica* in the third century. He used one Greek letter to represent the unknown, and other notations for squares and a different notation for cubes. Various symbols represented the fourth power, which he called square-square, the fifth power, which he called square-cubed. The cumulative effect proved to be effective as communication, but too wordy. However it provided the basis for the exponential notation we use today $(x, x^2, x^3, x^4, \ldots)$. The use of minus signs and addition signs did not emerge until the sixteenth century.

Teachers know the historical bases of mathematics and the problems societies faced that gave rise to mathematical systems. They incorporate the contributions made by individuals, both female and male, and of cultures, non-European and European, in the development of ancient and modern mathematics into their teaching. Through an understanding of the development of mathematics, they help students begin to see the practical applications and human side of mathematics.

<u>Mathematical World Views</u>

As the demographics of our country continue to change, teachers will face increasingly diverse students in their classes. Ensuring that all students will succeed in mathematics requires teachers who understand the influences of students' linguistic, ethnic, racial and socioeconomic backgrounds on learning mathematics. Teachers recognize that there are multiple mathematical world views, and are aware of their own mathematical world views and how they are similar to and different from the world views of their students.

For teachers whose families have spoken English for several generations and who have always lived in a society with a Greco-Roman-Western European mathematical heritage, it may seem as if there is only one mathematical world view. However, that perception is a false one. Mathematical world view is linguistically, culturally, and environmentally determined; vestiges of other world views persist in some families for several generations. Thus students may have a basic mathematical world view that is substantially different from that assumed by their teachers to be natural to students of their age level.

Examples of world view contrasts are myriad. They range from relatively subtle differences such as the French-speaking individual's viewing eighty as four score rather than as eight tens, or the British interpretation of 3.5 as three times five and 3 · 5 as three and five tenths rather than the reverse (as it is interpreted in the United States), to examples such as the following:

- The mathematics commonly taught in the United States uses base ten counting (except for computers, time, and measurement) but some societies use base-20 or base-60. Furthermore, many others incorporate place value into their counting (5234 is read five thousands two hundreds three tens four) and so do not understand preoccupation with place value.

- The English language incorporates "linear" logic (a implies b, c implies d, b and d together imply e; this process continues until a conclusion is reached); however, not all languages incorporate this style of reasoning. Other languages may use a "shrinking in" logic (all of the things bearing on a matter might be thought of as arrows arrayed in a circle around the problem; the circle of arrows is contracted until a conclusion is reached).

- The mathematics commonly taught in the United States describes geometric shapes statically. They are rigid objects described by characteristics such as number of sides, parallel or perpendicular sides, equal sides, angle type or size, and base shape; they occupy fixed space or move to another fixed space. However, some peoples view shape dynamically and/or topologically. They may see shapes as developing fluidly (a perspective which enables the Navajo to create symmetric and balanced shapes in their rugs as they weave them), or as being described primarily by relationship to reference points which do not lie along two perpendicular axes (north, south, east, west, the cardinal directions) but toward Denver, toward Salt Lake, toward Oklahoma City, toward Houston, might be more typical of the reference systems used in many cultures.

- Speakers of Indo-European languages read numbers left-to-right but sometimes use algorithms that require right-to-left operations. However, speakers of other languages reverse directions for one or both of these processes. Pushtu speakers, for example, read 256 as six hundred fifty two and Arabic speakers read the number U0, from right to left as fifty-six. A common way of writing twenty-one divided by three is seven is (indicated in the margin). And many cultures teach students to write addition horizontally, "regrouping" by looking ahead. They would add 459 + 666 within place value and would check ahead each time to see if any "carrying" would be required. They would write simply 459 + 666 = 1125.

  $$\frac{21|3}{7}$$

- The mathematics normally taught in the United States views fractions through a part-whole lens. A whole is divided into equal parts and some of those parts are considered (3/4 implies dividing 1 whole into four equal parts and taking three of them; 3/4 of a dollar implies considering 100 cents to be a whole, dividing the 100 cents into four equal parts of 25 cents and taking three of those - i.e. 75 cents). However, some languages do not even contain words for dividing things up (except maybe for a half); they build fractions up (take one object, quadruple it, and then take three of the resultant objects) and view fractional relationships proportionally (75 cents has the same relationship to $1.00 as three dollars have to four dollars).

As teachers' mathematical abilities become more sophisticated, they gain the ability to switch between mathematical world views and even to operate in two or more simultaneously. For students to develop the same abilities, teachers must help them to build on their *own* innate mathematical world views.

Mathematical Structures

Teachers who teach mathematics at any grade level understand the overall framework of mathematics. They view this framework as a coherent structure which both connects and transcends content strands and representations, rather than as isolated sets of rules, facts and formulae. They find this "big picture" to be critical to their understanding of the interrelationships and commonalities among topics such as arithmetic, algebra, functions, and geometry as well as to their ability to construct new meanings and make their own connections as they develop instruction.

Teachers view this overall framework through a variety of lenses. One lens focuses on the process and consequences of expanding mathematical systems. Expansion generally occurs because real problems cannot be solved within an existing set of constraints -- for example, an ice-plant worker who is only able to use positive integers cannot describe the temperature when it drops 10^{o} below 2^{o}C; he must expand his number system to include negative integers. In so doing, he gains the ability to solve the problem, adds 0 to his set of numbers, and retains the behavior of operations, but loses some convenient properties (adding and multiplying make bigger). Similarly, an electrical engineer who wishes to describe current cannot do so within the real number system; she must expand her number system to include the square roots of negative numbers (i.e. to the complex number system). Again, in so doing, she gains the ability to solve the problem and retains the behavior of operations. However, this time she loses a significant property, that of order.

A second lens permits examination of the effects of broad ideas (operations and/or properties) as they are applied to various systems. In some cases, systems prove to behave in essentially the same manner (to be isomorphic) - for example, computation with polynomials behaves exactly like computation with integers. In other cases, systems behave similarly in some respects but differently in others -- for example multiplication of integers is cumulative but multiplication of matrices is not.

A third lens permits the same object to be examined from different perspectives - seeing some of the same things but different ones as well. For example, a circle can be represented as a geometric object -- where the eye captures the smooth continuous behavior of the curve. It can be presented algebraically as $x^2 + y^2 = 9$ where the observer first sees the paired squared variables interacting symmetrically to create the circle. In a polar representation, $r = 3$, the focus is on the length of the radius (3) and the unimportance of the generating angle. The vector representation, $r = 3 \cos t + 3 \sin t$, highlights the generation of the circle by a rotating radius vector while the parametric representation, $x = 3 \cos t, y = 3 \sin t$, concentrates attention on the motion of a point around a circle.

A fourth structural lens involves the logical reasoning that takes place within a system. The existence of an axiomatic system enables you to prove conjectures within that system. For example, within the system of Euclidean Geometry, you can prove that the sum of the angles of a triangle is 180^{o} Changing the assumptions leads to different consequences. For example, an assumption essential to that Euclidean proof is that exactly one line can be drawn through a point parallel to another line. Assuming that an infinite number of lines can be drawn makes the sum of the angles less than 180^{o}, while assuming that no such line can be drawn (the geometry of the globe in which all lines -- great circles -- intersect) makes the sum greater than 180^{o}.

Teachers who teach of mathematics develop an understanding of how mathematical structure provides the foundation for content strands and how the different strands are linked together. An awareness of mathematical structure allows teachers to help students build a strong conceptual framework for mathematics.

The Role of Technology & Concrete Models in Mathematics

Understanding the history and the structure of mathematics is knowing that mathematics is not a static discipline. Even as you are reading this report, the uses of mathematics are changing. The search for solutions to new problems continues to create new mathematics. Teachers know the history of the development of mathematics and remain cognizant of the ever changing nature of the discipline. Many of the current changes in mathematics result from changes in technology. Teachers understand the changing ways in which we learn, teach and do mathematics. These changes are a part of their understanding of and approach to mathematics. NCTM Professional Teaching Standards state:

> *"Technology changes the nature and emphasis of the content of mathematics as well as the pedagogical strategies used to teach mathematics. Performing computational and representational procedures by hand is time-consuming, and students often lose sight of mathematical insights or discoveries as they become mired in the mechanics of producing the results. With the introduction of technology, it is possible to de-emphasize algorithmic skills; the resulting void may be filled by an increased emphasis on the development of mathematical concepts. Technology--computers and calculators--saves time and, more important, gives students access to powerful new ways to explore concepts at a depth that has not been possible in the past." (NCTM, p 134)*

Teachers who teach mathematics at any level integrate technology with mathematics in their approach to mathematics instruction. Technology makes some old mathematics unnecessary. Complicated calculations, repetitive graphing, long division can now be completed using calculators, allowing students and teachers to explore the mathematical ideas that underlie the mechanics of computation. The emphasis should be on using computers and calculators as tools to represent mathematical ideas and construct different representations of mathematical concepts. Computers are used to organize, represent and analyze data, to conduct simulations and to make predictions. Through the use of technology students can collect and organize data, such as recycling at the school, graph and interpret data and analyze patterns. Technology keeps the focus on the mathematical concepts not the laborious crunching of numbers. The use of technology also opens mathematical investigation to students at younger ages.

Teachers who teach mathematics at any grade level also integrate a variety of non-technological tools into their instruction to help students can explore and learn mathematics. Drawings, diagrams, analogies, and invented symbols provide a vehicle for connecting mathematical ideas. Teachers are able to use concrete materials, such as tangrams, base-ten blocks, fraction bars, attribute blocks, counters, etc., as models to help students develop mathematical concepts and solve problems. Through these tools, abstract concepts can be explored in a concrete way, allowing students to explore mathematics at an earlier age.

Mathematics instruction should encourage students to select appropriate tools from technology, concrete materials, drawings, and diagrams based on what they find most useful for a given problem. Teachers know when to introduce new tools that are appropriate for particular concepts or problems and that will expand a student's repertoire of problem solving tools.

Teaching Mathematics: Principles 2-10

Principles 2-10 present the pedagogical knowledge base for mathematics teaching. These principles are based on the recognition that subject matter cannot be separated from teaching and learning experiences. The principles describe what teachers know and do to create the kinds of learning communities in which all students learn mathematics in the ways envisioned by mathematics education reform and validated by research. As such, the principles take a particular stance on teaching mathematics, favoring active, learner-centered instructional paradigms; emphasizing the importance of students engaging in the critical processes that foster deep understanding; and recognizing the contribution that different kinds of knowledge can make to each instructional decision [made by teachers]. The principles honor, as well, teachers' roles as members of a professional community and as members of the larger community of stakeholders who can help support the learning of their students. An explication of each of the nine principles comes first, followed by four vignettes that illustrate their interaction with each other and with content in actual teaching.

Principle #2: ***Teachers who teach mathematics at any level understand how children learn and develop and can provide learning opportunities that support their intellectual, social and personal development.***

Teachers develop an understanding of research on how students learn mathematics and incorporate current theories and research from mathematics education into their instructional decision-making. They are aware of students' prior mathematical knowledge and help students link new ideas to prior learning. They are always cognizant of the importance of helping students make connections and develop a coherent framework for building an understanding of mathematics. Teachers select mathematical tasks that scaffold student learning of mathematics. In the selection of tasks, teachers consider the range of students' ages, abilities, interests, and experiences, recognizing the effects of these characteristics on learning.

Principle #3: ***Teachers who teach mathematics at any level understand how students differ in their approaches to learning and create instructional opportunities that are adapted to diverse learners.***

Teachers seek ways to affirm and support full participation and continued study of mathematics by ALL students. They develop an understanding of the influences of students' language, ethnic, racial, and socioeconomic backgrounds and gender on learning mathematics and incorporate this knowledge into their instructional decision-making. They know about areas of exceptionality in learning, including learning disabilities, visual and perceptual difficulties, and special physical or mental challenges and adapt instruction to address these special needs. Teachers of mathematics pose mathematical tasks that reflect a knowledge of the range of ways that diverse students learn mathematics and display a sensitivity to, and draw on, students' diverse background experiences and dispositions in their instruction. They select instructional materials and resources (e.g., manipulatives, visuals, hands-on problems) that reflect the life experiences, the variety of learning

styles and performance modes of their students. Teachers work to create learning environments in which students' diverse interests, linguistic, cultural, and socioeconomic backgrounds and special educational needs are respected, and in which the full participation and continued study of mathematics by ALL students is encouraged.

Principle #4: ***Teachers who teach mathematics at any level understand and use a variety of instructional strategies to encourage students' development of critical thinking, problem solving, and performance skills.***

Teachers value problem solving and reasoning as the basis for mathematical inquiry and use a variety of instructional strategies, such as questioning, tasks that elicit and challenge each student's thinking, and problem formulation, as ways to encourage critical thinking and problem solving. They vary their role in the instructional process, deciding when to provide information, when to clarify an issue, and when to let a student struggle with a difficulty, based on the content, the purposes of instruction, and the needs of the student. In the mathematics classroom, discourse in which students initiate problems and questions, make conjectures and present solutions, explore examples and counter-examples to investigate a conjecture, and rely on mathematical evidence and argument to determine validity provides the primary vehicle for developing a student's critical thinking in mathematics. Teachers also encourage the use of tools in the problem solving process: computers, calculators, and other technology; concrete materials, such as models; and graphic representations, such as diagrams, tables, and graphs.

Principle #5: ***Teachers who teach mathematics at any level use an understanding of individual and group motivation and behavior to create a learning environment that encourages positive social interaction, active engagement in learning, and self-motivation.***

Teachers create a learning environment that fosters the development of each student's mathematical power. They provide adequate time to explore and grapple with significant ideas and problems while providing a context that promotes the development of both mathematical skill and proficiency. They create a learning environment in which students' ideas and ways of thinking are respected and valued. Learning environments in mathematics classrooms consistently expect and encourage students to work independently and collaboratively, take risks by raising questions and formulating conjectures, and display mathematical competence through the validation and support of ideas with mathematical argument. Learning environments in mathematics classrooms are characterized by discourse in which teachers listen carefully to students' ideas and monitor student participation in discussions. Teachers use this information to decide when and how to encourage each student to participate. In the mathematics classroom students listen to, respond to, and question each another as well as the teacher.

Principle #6: ***Teachers who teach mathematics at any level use knowledge of effective verbal, nonverbal, and media communication techniques to foster active inquiry, collaboration, and supportive interaction in the classroom.***

Communication is at the heart of an inquiry based classroom. Teachers pose mathematical tasks that promote communication about mathematics and ask students to clarify and justify their thinking orally and in writing. One aspect of mathematical communication is learning how to attach mathematical notation and language to ideas. Teachers recognize the importance of the language of mathematics and develop students' abilities to use diagrams, tables, graphs, invented and conventional terms and symbols. They encourage the use of metaphors, analogies, and stories to support written or oral presentations of mathematical hypotheses, and explanations and arguments to foster active inquiry.

Principle #7: ***Teachers who teach mathematics at any level plan instruction based upon knowledge of subject matter, students, the community, and curriculum goals.***

Teachers plan instruction that represents mathematics as a discipline of interconnected concepts and procedures and emphasizes the connections between mathematics and daily living. Their instruction includes tasks that promote and discourse that extends student understanding of mathematical concepts, procedures, and connections. They select tasks that involve problem solving, reasoning, and communication as the basis for lessons and use discourse to develop their students' ability to reason and communicate mathematically. Teachers also plan instruction that promotes students' confidence, flexibility, perseverance, curiosity, and inventiveness in doing mathematics.

Principle #8: ***Teachers who teach mathematics at any level understand and use formal and informal assessment strategies to evaluate and ensure the continuous intellectual, social and physical development of the learner.***

Teachers observe, listen to, and gather information about students to assess what they are learning to ensure that each student is learning mathematics, to challenge and extend students' ideas, to adapt or change activities while teaching, and to make both short- and long-range plans. They use a variety of assessment methods, based on the development level, the mathematical maturity, and the performance mode of their students to determine students' understanding of mathematics. Teachers of mathematics select assessment methods that are consistent with what is taught and how it is taught. Teachers analyze individual students' understanding of mathematics, share this information with students, their parents, and other school personnel. They monitor their own instruction, making changes as necessary, based on what they learn from the assessments.

Principle #9: ***Teachers of mathematics are reflective practitioners who continually evaluate the effects of their choices and actions on others (students, parents, and other professionals in the learning community) and who actively seek out opportunities to grow professionally.***

Teachers examine and revise their assumptions about the nature of mathematics, how it is learned and how it should be taught, and experiment thoughtfully with alternative strategies in the classroom. Teachers observe and analyze a variety of approaches to mathematics teaching and learning as they focus on the tasks and discourse. They participate in workshops, courses, and other professional development activities specific to mathematics. Teachers analyze and evaluate the appropriateness and effectiveness of their teaching and reflect on learning and teaching both individually and with colleagues. Teachers participate actively in the professional community of mathematics educators (e.g., the National Council of Teachers of Mathematics and local affiliates) and discuss the ideas presented in professional publications with colleagues.

Principle #10: ***Teachers of mathematics foster relationships with school colleagues, parents, and agencies in the larger community to support students' learning and well being.***

Teachers participate in school, community, and political efforts to effect positive change in mathematics education. They build bridges to individuals and agencies within the communities that help them show the connections between mathematics and daily living. Teachers of mathematics work with parents and guardians as partners in the education of their students.

Vignettes

The four vignettes which follow have been created as examples of classroom instruction consistent with the INTASC standards for teaching mathematics. The vignettes are not meant to suggest that there is a singular "right" way to teach mathematics or the particular mathematical topics illustrated. As illustrations, the vignettes provide only very small samples of the kinds of mathematics instruction evident in the classroom of beginning teachers. They also illustrate only a small range of approaches that might be used by or performances that might be modeled by beginning teachers working within the framework of the standards. *Although the vignettes were intended to portray teaching rather than assessment episodes, they do illustrate the kinds of events a beginning teacher might attempt to capture to include in a collection of materials being assembled for assessment purposes.*

The first vignette, The Ducklings, shows how an elementary school teacher charges her class with the responsibility of determining how much food will be required to feed six ducklings. The students explore different counting strategies, patterns, and proportional reasoning in the process of feeding the ducklings. In the second vignette, Daily Earnings, Ms. Rodriguez shows how middle-school students' interest in money can provide the basis for investigating patterns and functions, and she emphasizes a variety of ways of representing these relationships. Ms. Morgan, who teaches a very diverse group of 9th graders, uses an apparent contradiction or miscount of protesters at a rally as a way of introducing students to strategies using decomposition of geometric figures and similar figures to estimate large quantities in the third vignette. Mr. Murphy redesigns a lesson for his Algebra 1 class to incorporate technology and real-world connections into the study of quadratics in the final vignette.

As you read through the vignettes, reflect on the INTASC principles that articulate effective teaching. Marginal comments are used to illustrate the link between the instruction and these principles. Each vignette includes annotations that clarify or describe the teacher's actions or reasoning as a way of focussing attention on critical aspects of mathematics teaching and making connections between the illustrations and the INTASC Principles.

Vignette One: The Ducklings

In this lesson, the teacher uses children's prior knowledge to build new conceptual understandings in new mathematical topics being introduced. The lesson conveys understandings of multiplication, the distributive property of multiplication and proportional reasoning through an interdisciplinary unit based in science. Proportional reasoning and the distributive property (e.g. $6 \times 24 = (6 \times 20) + (6 \times 4)$) have not been commonly introduced at lower grade. However, this teacher has become familiar with educational research that has found that connecting these concepts early empowers children later in mathematics. She decides to emphasize a deeper knowledge of content and the cognitive processes necessary to achieve these mathematical understandings.

She creates a situation that will be meaningful to students and builds from their prior knowledge as she helps them develop the new understandings. Her strategies include using physical representations (base 10 blocks), linking the visual/concrete model to both mathematical notation and the story situation, discussion in which she will assist students through her questioning to use problem solving and critical thinking.

The environment the teacher creates helps students feel comfortable participating and responding, seeing mistakes as opportunities to analyze and correct their thinking or work. It also provides opportunities for children at a wide range of achievement levels to be able to solve a problem successfully. Note the many ways children are able to solve the problem and how they are pushed to articulate their thinking. This approach, though initially time-consuming produces much deeper conceptual understanding among students, and develops critical thinking skills. This vignette primarily illustrates the following INTASC Principles:

> *Principle 2:* Teachers who teach mathematics at any level understand how children learn and develop, and can provide learning opportunities that support their intellectual, social and personal development.
>
> *Principle 4:* Teachers who teach mathematics at any level understand and use a variety of instructional strategies to encourage students' development of critical thinking, problem solving and performance skills.
>
> *Principle 10:* Teachers of mathematics foster relationships with school colleagues, parents and agencies in the larger community to support students' learning and well being.

Background information: The setting is an urban inner city school; a culturally diverse class of 2nd/3rd graders; 25% of students have not been successful in mathematics;

After the development of the concepts of reproduction, birth and growth of animals in her science unit, the teacher brought in a real incubation chamber with six duck eggs and related it to the concepts developed in the science unit. She then presented the following situational story (written on chart paper) to the class.

We have a big surprise this morning. We now have an incubation chamber and we have 6 duck eggs. Once our eggs hatch we will need to feed the ducklings. Each duckling will need an ounce of grain each day. So we will need a total of 6 oz. of grain each day to feed all the ducklings. How much food should we plan on having for the ducks for four weeks?

Teacher connects to students' experiences, and uses appropriate strategies (INTASC Principle 2) achieve learning goals (INTASC Principle 4)
connecting science and mathematics by creating story based on classroom animal unit

T: Can someone tell us what we're trying to find out in our problem?

Teacher promotes critical thinking and problem solving (INTASC Principle 4)

S: How much duck food we'll need.

T: Is there information in the story that can help us figure this out. Tamara?

TW: There are 6 ducks and we need 6 ounces of food a day.

T: Is that all we need to know? Kalem?

Teacher as facilitator to student's understanding of the problem (INTASC Principle 4)

K: We have to find out how much food we need for four weeks?

T: You've said we have 6 ducklings and we need 6 oz. of food for one day - how are we going to find out how much we need for four weeks?
K: We need to know how many days that is.
T: Who can tell us a way to find out how many days that is?
R: There are 7 days in one week. So for four weeks you add 7 + 7 + 7 + 7.
T: And how much is that?
R: 28
T: How did you get that?

Teacher elicits student thinking (INTASC Principle 2)

R: I did 7 + 7 is 14, plus 7 is 21, plus 7 is 28
T: Did anybody think about that in a different way? Brandon?

Teacher allows students to have choices in their strategies; demonstrates multiple ways to find an answer (INTASC Principle 5)
Teacher wants to increase the pool of available strategies to meet diverse needs of students.

B: I just multiplied 4 x 7 and got 28.
T: Sara?
S: I knew 7 and 7 was 14, and 14 + 14 is 28.
T: You're telling me we need to find out how much duck food we'll need for 28 days?
S: Yes.
T: Who has an equation we can use to solve this problem?
M: 6 x 28 = ? oz. each day.
T: What does the 6 represent?
M: That's 6 oz. of duck food.
T: And what is the 28?
M: That's the 28 days we're feeding them.
T: Work with your manipulative to solve our problem and record your solutions.

Teacher chooses materials for developmental needs (INTASC Principle 4)
teacher monitors and adjusts

Teacher circulates to monitor what students are doing and to pose leading questions for those who need them.

During her monitoring she assesses the solutions being developed and mentally selects (and sequences) solutions to be shared with the class when the whole group reconvenes.

As she circulates, she notices that Deon is adding 6 and 28.
T: Deon, explain to me what you're doing.
D: These (pointing to 6) are my duck food and these (pointing to 2 tens and 8 ones) are my 28 days.

Teacher assesses individual performance and adjusts instruction to meet student's needs (INTASC Principle 2)

T: Let's look at the story again.

The student rereads the story.

Concentrates on student developing understanding rather than just telling him.

T: What are we trying to find out?
D: How much food we need to feed the ducks for 28 days.

T: So show me how much food we need for 1 day. (Child shows 6 ones, representing 6 oz. of grain)
Now show me how much we need for 2 days.
(Child shows another group of 6)
O.K. What do you have on your work space now?

D: This (6 ones) is one day; this (other 6 ones) is the second day.

T: So, now you have 2 days; what do you need to show for 28 days?

D: Oh! 28 of these groups.

Teacher uses alternative explanations to assist student understanding (INTASC Principle 4)

The teacher continues to monitor how Deon proceeds as she continues to scan the rest of the class. The teacher notices that some students are beginning to complete the problem before others are done, so she suggests to those students to find an additional way to solve the problem and to record this new way, also.

Although on some days, the teacher has a number of groups or individuals record their solutions on the board and discuss them in turn, she decides to have the solutions shared today in a manner that will allow her to introduce an additional recording strategy that some students may wish to use.
Teacher evaluates learning goals and implements multiple strategies to promote critical thinking and problem solving (INTASC Principle 4)
Teacher purposely calls on Deon to make sure he understands.
Teacher makes provisions for individual students with particular needs (INTASC Principle 3)

T: Deon, would you share the way you figured out how much duck food we will need? Would you show us your paper?

Deon holds this paper up for the class to see, displaying 28 groups of 6 ones.

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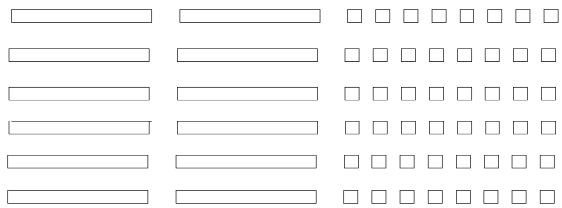
T: Would you explain what your paper shows?
D: This is all the food for the ducks.
T: How do you know that's all we need?
D: Because these are my 28 days (points to each of the groups).
T: So how much food is it that we need?
D: 166. No wait. 168 ounces of grain.
T: How did you figure that out?
D: I counted them.
T: Who wants to share a different way? Shannon.

Teacher creates classroom atmosphere where individual differences in thinking are respected. (INTASC Principle 3)

S: (Arranges 6 groups of tiles: 2 ten-rods and 8 units in each row.)



T: What does this stand for?
S: 6 ounces of food times 28 days.
T: Tell me why you set up your blocks this way.
S: Well, it's 6 ounces each day for 28 days so we could show 28 sixes but it's easier to show six 28's and it comes out the same.
T: How do you know that?
S: Well, it's like 2 threes is the same as 3 twos. They're both six.

Students articulate and justify their thinking.

T: O.K. So you have 6 rows with 28 in each row? Show me 28 in one of the rows.
S: Here are 2 tens and 8 ones. (pointing)

Teacher links new ideas to prior knowledge of student through inquiry (INTASC Principle 2)

Teacher asks the student, "How much is 2 tens?". (20). "So you're showing me 20 and 8." Following student direction, she writes under the equation:

6 x 28
6 x (20 + 8)

Prodded by teacher questions, Shannon explains that she figured out how much six 20's were and how much six 8's were, and added them together to get 168. The teacher records as Shannon explains:

Teacher stimulates student reflection and provides opportunities for active engagement. (INTASC Principle 2)

6 x 20 = 120
6 x 8 = 48
120 + 48 = 168

To verify this amount, the teacher first calls on Maria. Maria demonstrates six 20's with her manipulatives. She counts those for the class to prove it was 120. Sean is called on to prove that six 8's were 48. Sean skip counts by 8's to get 48 (i.e. 8, 16, 24, 32, 40, 48). The teacher calls on Tuan to get the total amount. Pointing to the rods, Tuan counts on from 120: 130, 140...160, 161, 162...168.

She reviews with the class that they decomposed (broke apart) the 28 into 20 + 8 so it was easier to work with, and multiplied each part of the 28 by 6. She outlines on the overhead with her marker the 6 x 20 area, removes the ten-rods, and asks what equation can be written to show the 6 groups of 10 rods.

Teacher develops a variety of representations for multiplication.

The equation is 6 x 20 = 120. This is written inside the tens part of the area model. She does the same for the ones part and writes 6 x 8 = 48. She then combines those numbers to find the total amount of duck food needed.

At this point the teacher focuses on the model she wanted to demonstrate as an additional strategy students might use. She uses this model because it connects multiplication to both algebra and geometry.

The teacher asks Shakana to share her strategy. This student used graph paper and outlined 28 groups with 6 in each group. When asked how she figured out the total amount of food she said she knew 10 sixes was 60 and then added another 60 to make 120. That was 20 rows. She then added 5 more groups of 6 to get 30. She now had 150. Then there were 3 more sixes which were 18. So 150 + 18 is 168.

Teacher continues to have students share alternate strategies. (INTASC Principle 4)

The teacher uses this model to introduce a proportion table to students.

T: When Shakana looked at her rows, what did she count by? No response. So teacher asks: How many are in each row?
S: Six.
T: So what are we counting by?
S: Sixes.
T: So in one row how many do we have?
S: Six.

Teacher begins to record a table that has two columns labeled days and ounces:

DAYS	OUNCES
1	6

T: One day has 6 ounces. How many in 2 days?
A: 12 ounces.
T: And in 3 days?
A: 18 ounces.

Teacher records on the proportional table as she discusses this with the students:

DAYS	OUNCES
1	6
2	12
3	18

The teacher continues this way until she and the class reach 10 days and 60 ounces. She then asks:

Teacher stimulates students to link current knowledge to new ideas. (INTASC Principle 2)

T: If we know 10 days, could we figure out 20 from that?

No response from the students leads teacher to pose the following question:

T: How do you get from 10 to 20, Richard?
R: Double it.
T: What do you do to double something?
R: You multiply by 2.
T: We know 10 days doubled is 20 because 10 x 2 = 20, how do we know how many ounces in 20 days?
S: Multiply the 60 by 2.
T: How much is 2 x 60?
S: 120 ounces.

Teacher continues recording on the proportional table:

DAYS	OUNCES		DAYS	OUNCES
1	6		6	36
2	12		7	42
3	18		8	48
4	24		9	54
5	30		10	60

T: We have 20 days and we need to find the number of ounces for 28 days. What's left to do?
K: Find 8 more days.
T: And how many ounces are there in 8 more days?
S: 48 ounces.

Teacher instruction addresses cognitive levels of students. (INTASC Principle 2)

Teacher asks several children how they knew 48 ounces was needed for 8 days. Some children have multiplied 8 x 6 is 48. Other students find on the table 8 days and 48 ounces. The teacher records the following lines off to the side of the proportional table:

These last lines have been added in order to make it very clear to students that they are adding the number of ounces for 20 days and 8 days to find the total amount of ounces for 28 days.

20	120
8	48
28	168

The teacher asks if anyone has a different way of finding the solution because she noted that one student, Marcia, had used a compensation strategy. Marcia stated that she likes working with easy numbers, so she used 6 x 30. She knew that was 180. That was too much, because there were 2 extra days. So she had to take away 12 ounces from the 180 to get 168 ounces for the 28 days.

Teacher provides development of critical thinking skills.

The teacher ends this lesson by reviewing each solution and connecting them to the story situation. Her summary shows that multiple strategies can get the same, correct answer and that the students now have other ways of finding solutions to problems (proportional table and area model). She emphasizes that the different solutions grew out of students being able to decompose numbers (break them apart) so they are easier to work with.

Teacher reviews important concepts from the lesson to reinforce new knowledge and to make the mathematical connections explicit.

Homework assignment: The teacher wants to make a connection to the home and wants the parents to feel connected to the school. She develops a story situation similar to the one used in the day's lesson so the children will be familiar with the structure of the problem. She understands the importance of making parents feel comfortable with the homework assignment so she couches the story in a situation that is familiar to parents as well.

HOMEWORK

Directions to Parents: Please help your child solve the following problem. *Remember, it is important to allow your child to decide how to solve the problem. Allow your child to think about this.*

HOW MUCH DOES ALL THE MILK COST THAT YOUR FAMILY USES IN A WEEK?

(**Note:** There are 4 cups in a quart and 8 cups in a 1/2 gallon. For this lesson, assume that 1 glass = 1 cup.)

You may want to use the following questions after your child has thought about how to solve the problem:

(1) What are we trying to find out?
(2) What information do we need in order to find that out?
(3) What do we need to know about how our family uses milk?
(4) How does what we use in a day help is know how much we use in a week?
(5) Milk is not sold by the cup (glass). How can we figure out how many containers of milk we use?
(6) We know how much milk the family uses for the week. What else do we have to find out about the milk?

Please tell your child to draw a picture or chart as part of the solution. You should also show the solution using numbers. *There is more than one way to solve this problem. You and your child may show two different ways to solve the problem.*

Summary: The teacher supported the different levels of development within her class. Deon who functions at a concrete level and Shannon who operates at a more abstract level were able to succeed through the use of multiple teaching and learning strategies. She also varied her role by being instructor, facilitator, and coach. She developed multiple representations of multiplication; concrete representations, pictorial representations, the repeated addition model, the area model and the distributive model. Her constant monitoring allowed her to catch Deon's misconception and assist him. As the teacher grows in experience, she will more closely model the practices observed in accomplished teachers of mathematics. She will make more connections back to the original, real-world problem as students explore different forms of representation. She will make more explicit connections of computational mathematics to other areas and to numerical estimation and number sense.

Vignette Two: Daily Earnings

In the following vignette, Ms. Rodriguez, uses her students interest in earning money as a way of motivating them to investigate functions. The vignette illustrates how the teacher uses large group, small group, and individual work to create a classroom environment that encourages students to engage in learning. Her emphasis on problem solving and reasoning show the value of critical thinking in her classroom. The vignette also illustrates how students make sense of mathematics through verbal communication and the importance of using mathematical language and notation to foster inquiry. Finally, Ms. Rodriguez models the use of informal assessment techniques to monitor learning within the classroom and uses the information she gathers to reflect on and modify her own instruction.

The vignette primarily illustrates the following INTASC Principles:

> *Principle 4:* Teachers who teach mathematics at any level understand and use a variety of instructional strategies to encourage students' development of critical thinking, problem solving, and performance skills.
>
> *Principle 5:* Teachers who teach mathematics at any level use an understanding of individual and group motivation and behavior to create a learning environment that encourages positive social interaction, active engagement in learning, and self-motivation.
>
> *Principle 6:* Teachers who teach mathematics at any level use knowledge of effective verbal, nonverbal, and media communication techniques to foster active inquiry collaboration, and supportive interaction in the classroom.
>
> *Principle 8:* Teachers who teach mathematics at any level understand and use formal and informal assessment strategies to evaluate and ensure the continuous intellectual, social and physical development of the learner.

Ms. Rodriguez teaches middle school mathematics in a small, rural community. Her heterogeneously grouped mathematics classes are mainly comprised of an equal number of students from the three major ethnic groups of the community - African American, Hispanic, and White - and a few Asian students. The students know each other well. Most of them have gone to school together all of their lives, are on the same athletic teams, and are in and out of each other's homes. They are a very social group and enjoy each other's company. Many of the students work part time and earn money by babysitting, mowing lawns, or doing other jobs in the community.

After reading her students' journals from the previous week, Ms. Rodriguez discovered that the two-day lesson she presented last week on patterns and functions was not well received by her students. Journal comments suggested that students were bored and had a difficult time grasping the concepts.

Ms. R. realizes that she must design a lesson that will engage students. She routinely uses observations, journals, and questioning strategies to monitor learning and makes adjustments as needed.

Taking into account the students' background, she realized that the lesson must capture their interests and get them involved. Ms. Rodriguez planned a new lesson on patterns and functions. She wanted students to understand that the basic concept of functions is that two quantities are in some way related. The value of one quantity may depend on the value of another quantity. The activities of her new lesson will include students investigating patterns using a table, describing patterns using algebraic notation (e.g., y = D x 100), and representing them graphically.

She consulted with other teachers and reviewed resource materials, including problem solving activities and the NCTM Addenda series, in search of ideas to present the concepts in a more interesting way. Her team leaders suggested that she find a context for the problem solving that will capture her students interest, so they won't necessarily see that the lesson is mathematics. Ms. Rodriguez knew that all of her students are interested in earning money, so she decided to work with tasks related to making money. The following vignette describes how the lesson went.

Ms. Rodriguez frequently collaborates with other teachers and considers students' interests and needs in planning instruction.
Teachers are reflective practioners who continually evaluate the effects of their choices and actions and who actively seek out opportunities to grow professionally. (INTASC Principle 9)
Teachers provide learning opportunities that support students' intellectual, social and personal development. (INTASC Principle 2)

When Ms. Rodriguez began class the following day, she gave her students the following task:

> You are being hired to work as my assistant for the next 20 days. Would you prefer to be paid:
> - $ 100.00 per day
>
> OR
> - $.01 for the first day, $.02 for the second day, $.04 the third day, $.08 the fourth day and so on ?

The class immediately began discussing the question posed, and some students offered conjectures. Ms. Rodriguez led a class discussion asking for possible strategies for answering the questions. One student suggested that the information could be organized in two tables, one for each scenario. After more discussion, the class decided to combine the two into one table so that comparisons could be made more easily. The class discussed the different entries needed in the table and came up with a format as shown in the following table.

Ms. Rodriguez values problem solving and reasoning as the basis for mathematical inquiry.
Teachers understand use a variety of instructional strategies to encourage students' development of critical thinking, problem solving, and performance skills. (INTASC Principle 4)

Day	Option 1		Option 2	
	Day	Total	Day	Total
1				
2				
3				
4				

After 10 minutes of discussion, Ms. Rodriguez senses that some students were tuning out, so she decided to organize the students into cooperative learning groups to complete the investigation. The students quickly organized themselves into pre-assigned groups according to the month's group schedule developed by Ms. Rodriguez. She assigns students to groups on a monthly basis to ensure that students interact with different peers and that groups are balanced in regards to ethnicity, gender, and ability.

Ms. Rodriguez uses a variety of grouping techniques including pairs, cooperative learning groups, whole class and individual, to promote a positive learning environment and to ensure full participation of all individuals.
Teachers use an understanding of individual and group motivation and behavior to create a learning environment that encourages positive social interaction, active engagement in learning, and self-motivation. (INTASC Principle 5)

Ms. Rodriguez circulated among the groups, mainly observing as students worked to complete the table. Calculators were used in several groups as students computed the second option. She observed that one group is not using the calculator to determine the total money earned per day for the second scenario. She jotted down a note to ask that group why.

Ms. R. records her observations regarding the different techniques groups are using to solve the problem.
Teachers use formal and informal assessment strategies to evaluate and ensure the continuous intellectual, social and physical development of the learner. (INTASC Principle 8)

When Ms. Rodriguez saw that most groups were finished, she decided to refocus the groups for a whole-class discussion. She brought students back together and led a discussion on the groups' findings. One group reported that the second scenario pays off better in the end than the $100 a day. The other groups concurred. Ms. Rodriguez asked a different group to show how they came to that same conclusion. They submitted the following table:

Day	Option One	Total	Option Two	Total	Day	Option One	Total	Option Two	Total
	$ 100 / day		**1,2 ,4 8 ...**			**$ 100 / day**		**1,2,4,8 ..**	
1	$ 100	$100.00	$0.01	$0.01	11	$ 100	$1,100.00	$10.24	$20.47
2	$ 100	$200.00	$0.02	$0.03	12	$ 100	$1,200.00	$20.48	$40.95
3	$ 100	$300.00	$0.04	$0.07	13	$ 100	$1,300.00	$40.96	$81.91
4	$ 100	$400.00	$0.08	$0.15	14	$ 100	$1,400.00	$81.92	$163.83
5	$ 100	$500.00	$0.16	$0.31	15	$ 100	$1,500.00	$163.84	$327.67
6	$ 100	$600.00	$0.32	$0.63	16	$ 100	$1,600.00	$327.68	$655.35
7	$ 100	$700.00	$0.64	$1.27	17	$ 100	$1,700.00	$655.36	$1,310.71
8	$ 100	$800.00	$1.28	$2.55	18	$ 100	$1,800.00	$1,310.72	$2,621.43
9	$ 100	$900.00	$2.56	$5.11	19	$ 100	$1,900.00	$2,621.44	$5,242.87
10	$ 100	$1,000.00	$5.12	$10.23	20	$ 100	$2,000.00	$5,242.88	$10,485.75

The reporter for that group explained that figuring the total for the $100/day scenario was easy, but that figuring the total for the other situation was a little more complicated and that they used the calculator to add up all of the subtotals for each day to figure the cumulative total.

Ms. Rodriguez varies her role in the instructional process (e.g., instructor, facilitator, audience) and uses questioning to access students' thinking.
Teachers use a variety of instructional strategies to encourage students' development of critical thinking, problem solving, and performance skills. (INTASC Principle 4)

Ms. Rodriguez questioned the group that did not use the calculator to determine the cumulative total. A student from that group explained that at first they were adding up all of the subtotals, but they saw a pattern and realized that all they had to do was add the next day's pay to the previous day's total. He illustrated this method on the table. Another student explained that she saw another pattern.

Ms. R. uses her notes from earlier observations to call on a group that discovered a pattern in the table that allowed them to solve the problem without using the calculator.
Teachers use formal and informal assessment strategies to evaluate and ensure the continuous intellectual, social and physical development of the learner. (INTASC Principle 8)

Jana: If you look at the last column, all you have to do is double the previous day's total and add 1.

Harry: Yeah, I get it. 31 x 2 = 62 and 62 + 1 = 63.

Leah: The doubling makes sense because you're doubling your salary each day. In fact, another way to figure it is to double your salary for the day and subtract 1. You know, 32 x 2 = 64 and 64 - 1 = 63.

Ms. R: What do you mean by 1 - one dollar or what?

Ms. Rodriguez questions a student in order to clarify the student's thinking and to make sure other students are making the same assumption.

Leah: No, one cent.

Based on the students' explanations, Ms. Rodriguez writes the following expressions on the board:

double the previous day's total and add 1 cent

2 x 31¢ = $.62

$.62 + .01 = $.63

double your salary for the day and subtract 1 cent

2 x 32¢ = $.64

$.64 - .01 = $.63

Ms. Rodriguez models effective communication strategies by restating ideas using mathematical notation.
Teachers use effective verbal, nonverbal, and media communication techniques to foster active inquiry, collaboration and supportive interaction in the classroom. (INTASC Principle 6)

Sami: I see it for the smaller numbers, but does that work for the larger numbers too?

Students decided to verify these two methods using their calculators. Afterwards, Ms. Rodriguez asked students if they see any patterns in the $100/day scenario.

Ms. Rodriguez senses that students understand the patterns involved with these two methods and focuses their attention on the other scenario. She asks questions to stimulate discussion and probe for learner understanding.
Teachers use formal and informal assessment strategies to evaluate and ensure the continuous intellectual, social and physical development of the learner. (INTASC Principle 8)

Lou: That's easy. The total increases by $100 each day. You can just count by hundreds.

Dara: Another way is to multiply the day by 100. You know, 3 times 100 is 300, 4 times 100 is 400, and so on.

Kelly: You can put a multiplication sign between the first two columns and an equal sign between the next two columns or write day x 100 = total pay.

Ms. R: Using the information we now have about the first three columns, I want you to work in your groups to answer the following questions: What is the relationship between the day and the total? What is a rule to determine the total earned from the day? Write an algebraic expression to represent this relationship.

Throughout the lesson, Ms. Rodriguez takes on different roles in relation to what's going on in the classroom and in response to students' needs. Ms. Rodriguez uses student responses to guide her instructional decisions such as grouping, questioning, and next steps.

Groups worked on the assignment for a short time and then reported their findings. Based on the group reports, Ms. Rodriguez recorded the expression, D x $100, in the table. Realizing that the bell was about to ring, she posed the following questions for students to think about for tomorrow's lesson: How can you write an algebraic expression in terms of D to represent the other relationship? She offered a hint to students to think about powers of two. What do you think the graphs of these two relationships would look like?
Students left class talking about these questions.

Ms. R. sets the stage for tomorrow's lesson in which students will see another way of representing a relationship.
Teachers use a variety of instructional strategies to encourage students' development of critical thinking, problem solving, and performance skills. (INTASC Principle 4)

Vignette Three: Photographs

In the following vignette, a beginning teacher demonstrates how her knowledge of diverse learners, and in particular the influences of students' linguistic, ethnic, and racial backgrounds on learning mathematics, affects the instructional decisions she makes in her classroom. Ms. Morgan shows a variety of ways to pose questions and tasks that elicit and challenge each student's thinking and encourage problem solving and mathematical reasoning. Throughout the vignette she observes, listens to, and gathers information about her students to assess what they are learning, she also designs assessments that are aligned with what and how she is teaching.

This vignette primarily illustrates the following INTASC Principles:

> *Principle 3:* Teachers who teach mathematics at any level understand how students differ in their approaches to learning and create instructional opportunities that are adapted to diverse learners.
>
> *Principle 4:* Teachers who teach mathematics at any level understand and use a variety of instructional strategies to encourage students' development of critical thinking, problem solving, and performance skills.
>
> *Principle 8:* Teachers who teach mathematics at any level understand and use formal and informal assessment strategies to evaluate and ensure the continuous intellectual, social and physical development of the learner.

Ms. Morgan is a first year teacher who has completed an internship program and is currently working with a veteran teacher as part of a mentor program. She teaches 9th graders in an integrated mathematics curriculum. Her class is composed of 33 students with a variety of primary languages. One-third of the students do not speak English well. The students have been studying similar geometric figures. The following vignette describes a geometry lesson in which students will explore the relationship between perimeter and area of similar figures and the ways in which similarity of figures can be used to estimate areas.

As students enter the classroom, they pick up a copy of a task sheet which includes steps they are to follow. The students immediately settle into their pre-assigned groups of four. The task described on the sheet is as follows:

Ms. Morgan knows that language learners have difficulty following oral directions, so all tasks are also presented in writing.

Ms. Morgan has learned that random assignment of groups can lead to a mixture of students that will interfere with learning; for example, grouping three native speakers of Spanish with one of Vietnamese may result in the Vietnamese speaker being excluded from the discussion; assigning an Arab girl to a mixed sex group may cause her to withdraw because Arab girls do not interact with boys who are not family members.

After a recent demonstration at the Capitol in Washington D.C., the newspaper headline read, "15,000 People at Rally". However, the article stated that demonstration organizers claimed 30,000 people were present. Using the attached (12"x16") aerial photograph of the rally which claim do you believe is more accurate and why? How did the demonstration organizers come up with their estimate? How did the headline writer come up with a smaller estimate?

Ms. Morgan begins with a discussion of the importance of the issue because she knows that in a class with many language learners, students' views of why something is important may not reflect what is culturally significant in the United States. She takes care to permit students to bring their own experiences into the response.

Ms. Morgan begins the class by asking groups to spend two minutes discussing why this issue might be important. She then asks several groups to report their findings to the whole class.

Teachers create instructional opportunities that are adapted to learner diversity. (INTASC Principle 3)

Before tackling the problem of estimating the number of demonstrators, Ms. Morgan reviews the concept of similarity of figures with her students, and has them investigate the relationship between perimeter and area of similar figures. She has the class begin with rectangles, similar to the shape of the photograph of demonstrators, and distributes a different size rectangle to each group of students.

Ms. Morgan has defined the task very closely because she wants to be sure that all students understand what they are to do and that they can apply their knowledge. When working with language learners, it is often necessary to define the task more closely than would otherwise be the case. Students who are unsure of their English may be unable to follow imprecise directions.

Teachers create instructional opportunities that are adapted to learner diversity. (INTASC Principle 3)

The Relationship Between The Perimeter and The Area of Similar Figures

1. Find the length, width, perimeter, and area of the figure your group received.
2. Draw a figure similar to the original with sides 1/2 as long. Find the length, width, perimeter, and area of this figure.
3. Draw a figure similar to the original with sides 1/4 as long. Find the length, width, perimeter, and area of this figure.
4. Draw a figure similar to the original with sides twice as long. Find the length, width, perimeter, and area of this figure.
5. Draw a figure similar to the original with sides four times as long. Find the length, width, perimeter, and area of this figure.

Figure	Length	Width	Perimeter	Area
regular				
1/2				
1/4				
twice				
four times				

What happens to the perimeter as when you double the sides? What happens to the perimeter when you make it four times as large?

What happens to the area when you double the sides? What happens to the area when you make it four times larger? Do you see a pattern?

Working in groups, students are to measure the lengths of the sides of the figure and then to draw figures similar to the original figure but with sides one half as long, one fourth as long, twice as long and four times as long. The students then are to find the perimeter and the area of each figure and find the ratios of those quantities. Each group has access to dot paper, graph paper, square tiles, and geometry software that support a variety of approaches to developing solutions to the problems.

Ms. Morgan has provided students with options for carrying out the task but not for shaping the task themselves.
Teachers plan instruction based upon knowledge of subject matter, students, the community, and curriculum goals. (INTASC Principle 7)

As the students begin to work on the task, Ms. Morgan circulates around the room monitoring what the groups are doing as well as asking leading questions or giving hints to groups that are stuck.

She stands back to observe how the groups are progressing. One group is busy at the computer and she nods approvingly at their energy. A second group is just sitting and seems to be discussing something other than the mathematics task. She walks over and asks the group, "What have you done so far, how are you approaching the problem?"

Ms. Morgan continues to monitor how groups are progressing by observing, listening to, and gathering information about students to assess what they are learning. She redirects groups as necessary.
Teachers use formal and informal assessment strategies to evaluate and ensure the continuous intellectual, social and physical development of the learner. (INTASC Principle 8)

Kara: We're done.

Ms. Morgan: Well, how did you get the answer?

Evan: Kara told us what to write.

Ms. Morgan: Can you explain how you got your answer?

Evan: I'm not sure.

Ms. Morgan: I want you to talk through the problem so that when I return any of you can explain to me how you are approaching the problem.

In the next group, Ms. Morgan finds that everyone is working but Juan. She asks him what is wrong, but gets no answer.

Lourdes: Juan doesn't understand what he is supposed to do.

Ms. Morgan turns to Juan (and speaks clearly), "Is that true, Juan? Do you not understand the instructions?"

Ms. Morgan understands that conceptual learning is the same no matter what the student's language, and her goal is for the students to learn the mathematics; she does whatever she can to facilitate that learning.
Teachers create instructional opportunities that are adapted to learner diversity. (INTASC Principle 3)

Juan: Everyone is talking too fast and at the same time. I can't understand.

Ms. Morgan: Would it help if Lourdes explains what they are doing in Spanish?

Juan: Yes, that would be good.

As Ms. Morgan leaves the group, Lourdes takes the worksheet and pencil and starts working with Juan. The other two group members keep on working on the task. Ms. Morgan goes to the next group.

Ms. Morgan: Josef, would you please explain your group's solution to me."

Josef: Well...uh...uh...uh...uh...we don't exactly have one - but we know how to do it!

Ms. Morgan: I'd like you to do a solution that you can present to the rest of the class. There are several ways to approach this problem and I'd like the class to be able to compare them and talk about the advantages of each.

Ms. Morgan gets the group back on task by getting them to invest in the process.
Teachers use an understanding of individual and group motivation and behavior to create a learning environment that encourages positive social interaction, active engagement in learning, and self-motivation. (INTASC Principle 5)

Evan: That sounds interesting. We didn't do *that* last year.

The next group is on task, so Ms. Morgan goes over to one where the two boys are working together and the two girls working separately -- it looks as if the group has divided into two parts.

Ms. Morgan: Show me your shapes and tell me what you found.

Khanh: We find perimeter ratios same and area ratios squares.

Ms. Morgan ignores Khanh's English-learner grammar. She understands him well enough to understand what the group has learned. She decides to use questions to find out whether the predominantly Vietnamese group has split by sex for cultural reasons but maintains communication or whether it has actually split.
Teachers create instructional opportunities that are adapted to learner diversity. (INTASC Principle 3)

Ms. Morgan: Do you think that your observation is always true? What would be the perimeter and area if the side was 10 times as long?

Khanh and Trang (one of the girls) together say, "Perimeter 10 times and area 100 times."

Ms. Morgan knows she must use different strategies with different groups, sometimes providing information, other times clarifying an issue, and often letting a group struggle with difficulties to encourage the development of critical thinking and problem solving abilities.
Teachers use a variety of instructional strategies to encourage students' development of critical thinking, problem solving and performance skills. (INTASC Principles 4)

Ms. Morgan: What if it was only 0.1 as long?

Trang and Mei: Easy -- perimeter .1 times and area .01 times.

Ms. Morgan: Why? How could you find out if your answer is correct?

Trang: We could measure.

Ms. Morgan: Do you think there is a rule?

All four heads nod yes.

Ms. Morgan: Why?

Ms. Morgan brings the class together and asks several groups to report on their findings. She picks Terrence's group first. Terrence gets up to the overhead and, using dot paper (plastic), draws the rectangular shape given to his group, a 10 by 15 cm rectangle. To show what happens when the measures are doubled, he draws a 21 by 31 cm rectangle and then displays the table the group constructed for the ratios of the sides, the perimeters and the areas. Puzzled by the obviously incorrect lengths, Ms. Morgan asks the group to explain the source of the numbers.

Ms. Morgan: How did you get your answers?

Terrence: We counted the dots (e.g. 11 dots on one side) to find the measure of the sides and just doubled each count (e.g. 22 dots, which would result in a side 21 cm long).

She is uses questioning both to help the group find its own error and to understand, herself, how the error was made.
Teachers use effective verbal, nonverbal and media communication techniques to foster active inquiry, collaborative and supportive interaction in the classroom. (INTASC Principle 6)

Ms. Morgan: Show me how you counted to get the answer.

Many of the students identify the problem with Terrence's reasoning and explain: "You're supposed to count the number of spaces between the dots, not the number of dots."

When Terrence turns to Ms. Morgan for a confirmation of the comment, she asks, "What do you think?"

Terrence (after a brief conference with the others in his group) says, "They're probably right, 11 by 16 seemed kind of funny. Then the double is 20 by 30."

Ms. Morgan is using questioning both to assess what the group understands and whether they can evaluate their own reasoning.

After three groups report their results, the class discusses their outcomes. Lourdes notices that when the sides were doubled the ratio of the perimeters was 2:1 for each group and that the ratio of the areas was 4:1 for all groups and asks if anyone else saw that.

Ms. Morgan encourages the students to clarify their ideas orally and to challenge one another's ideas.
Teachers use knowledge of effective verbal, nonverbal and media communication techniques to foster active inquiry, collaboration and supportive interaction in the classroom. (INTASC Principle 6)

Juan: We all started with the same rectangle so it's obvious that we should all get the same answer."

Melissa: No, we each had a rectangle with different dimensions, see the perimeters and areas aren't the same, just the ratios. What does that mean, Ms. Morgan?

Ms. Morgan lets the students continue to look for patterns, knowing that their discussion, with a little prompting will lead them to a generalization about the relationship between the ratio of the sides and the areas of similar figures.

Kara notices that the ratios are also the same for each group when you quadrupled the sides, the ratio of the sides was 4:1 and the ratio of the areas was 16:1.

Ms. Morgan promotes discourse in which students initiate problems and questions, make conjectures, and explore examples as vehicles for establishing the validity of arguments.

Ms. Morgan: Can you make any statements that are always true for finding the relationship of perimeter and area of similar figures?

Teachers use a variety of instructional strategies to encourage students' development of critical thinking, problem solving and performance skills. (INTASC Principle 4)

Evan: Whenever you double the sides of a figure the area is quadrupled and whenever you quadruple the sides, the area is 'sixteenupled' or whatever multiplied by sixteen is!

Teachers of mathematics value reasoning as the basis for mathematical inquiry and use questioning to challenge student thinking and to encourage mathematical reasoning.

Ms. Morgan: What do you think about that Keisha?

Teachers use a variety of instructional strategies to encourage students' development of critical thinking, problem solving and performance skills. (INTASC Principle 4)

Keisha: Well, it looks right so far, but we've only been working with rectangles, so maybe it's only true for rectangles.

Josef: Let's try it with triangles.

Ms. Morgan: O.K. I want everyone to draw a right triangle with sides of 9 cm, 12 cm, and 15 cm. Find the perimeter and area. Then draw a similar triangle with sides 1/2 as long as the original and find the ratio of the sides, the perimeters, and the areas.

Ms. Morgan starts to walk around and observe how students are solving the problem. She notices that Kim is done and wrote down the following ratios, 1:2 for perimeter and 1:4 for the area, without actually finding the perimeter and the area. The other members of her group, however, drew the triangle and are calculating the perimeter.

After a few minutes, Ms. Morgan asks the class what ratios they found. All agreed that the relationship for when you doubled the sides of a rectangle, also worked for a triangle.

Ms. Morgan then drew the following table on the overhead:

sides	2x	3x	4x	5x
perimeter	2:1	3:1	4:1	5:1
area	4:1	9:1	16:1	25:1

Ms. Morgan: Do you see any patterns?

Maya: The perimeter ratio is just the same as the ratio of the sides, if you double the sides, you double the ratio.

Ms. Morgan: That's true, and what about the number in the ratio areas, 4, 9, 16, 25 ...?

Juan: Aren't they squares?

Lourdes: You mean the square of the number is the ratio of the sides?

Ms. Morgan: That's what it looks like to me. Can anyone express the relationship in words?

Keisha: It looks like the ratio of the perimeter is always the same as the ratio of the sides and the ratio of the areas is the square of the ratio of the sides.

Ms. Morgan: Great, Keisha. So what does that all mean Evan?

Evan: It means you tricked us into doing a lot of computation of perimeters and areas that we didn't have to do! But you won't trick us again!

Ms. Morgan: That may be true, and you've learned another theorem. Write it down in your notebooks. We may get a chance to use it in the demonstration problem.

Ms. Morgan refocuses the class on the crowd estimation problem. She gives each group a small rectangular piece of acetate and asks them to put the shape over some part of the crowd and count the number of heads.

Ms. Morgan: How can we use the idea of similar figures to approximate the total number of people in your picture?

Kara: What does this have to do with area?

Ms. Morgan knows that if she answers Kara's question, she would be removing the thinking from the activity. She monitors students responses and adjusts to keep a critical thinking focus.
Teachers use a variety of instructional strategies to encourage students' development of critical thinking, problem solving, and performance skills. (INTASC Principle 4)

Ms. Morgan: Is there a connection between the area and the number of heads? I want each group to consider that question as you work.

The groups get started, Ms. Morgan stands back and observes. She notes that only one student in each group is actively working -- and makes a mental note that next time she probably should be sure each student has an acetate shape so they can all count and then compare their findings within their group rather than just across groups as she has planned for today.

Ms. Morgan evaluates the appropriateness of her instruction and plans adjustments based on this reflection.
Teachers are reflective practioners who continually evaluate the effects of their choices and actions on others and who actively seek out opportunities to grow professionally. (INTASC Principle 9)

Ms. Morgan pulls the class back together and asks, "Kara, what did your group decide about your question?"

Kara: We couldn't figure out exactly what the connection was.

Terrence: Our group thought that if you think about graph paper, you can think about one head to a square. This means that the area is about the same as the number of heads.

Ms. Morgan knows that, although the process may take more time, more learning occurs when the students provide the ideas than when she does. She creates a climate in which students engage actively in learning.
Teachers use an understanding of individual and group motivation and behavior to create a learning environment that encourages positive social interaction, active engagement in learning, and self motivation. (INTASC Principle 5)

Ms. Morgan: What about other groups, what did you find?

Several groups chorus: We found the same thing.

Ms. Morgan asks each group to use its understanding of ratio and area to estimate the size of the crowd. After they have their estimate, they should write their estimate and the equation they used to get the estimate on a piece of butcher paper taped to the front wall of the room.

Group	*Acetate*	*People/ Acetate*	*Equation*	*Estimate*
1	1" x 1"	123	12 x 16 x 123	23,616
2	3/4" x 1"	102	102 people x 16 x 16	26,112
3	2" x 2"	401	401 x 48	19,248
4	1" x 1"	95	12 x 16 x 95	18,240
5	3/4" x 1"	112	ratio of sides = 1:16; ratio of areas = $1:16^2$, so 112 x 16^2	28,672

Ms. Morgan brings the class back together and asks, "Should each group have the same estimate."

Sean: Yes, because there was one accurate number of people at the rally.

Hameed: That's true, but our numbers are estimates so they'll be different.

Sean: Oh yeah, but they should still be close.

Ms. Morgan: Good point, so what could we do to use these estimates to get an even better estimate?

Hameed: We could average the estimates and come up with one number.

Kara: What does this have to do with the theorem you made us write down?

Ms. Morgan: Did anyone use the new rule you discovered about the ratio of the sides and the ratio of area of similar figures?

Kara: Oh no, did we have to?"

Juan: Ms. Morgan, we didn't use it, but we came up with a good estimate anyway. Isn't that OK?

Ms. Morgan: I just asked if anyone made any connections.

Keisha: Ms. Morgan, our group saw that the acetate figure was ¾" x 1" and was similar to the photograph and the ratio of the sides was 16:1 so we knew that the ratio of the areas was the square of 16. We did use it in our computation!

Evan: We had the same size acetate, and our estimate was close to theirs. We noticed that the acetate would make 16 rows and 16 columns on the big picture. We didn't use the rule but we still got a good answer. Did we all have to solve it the same way?

Lourdes: I think Evan's right. You always say we can solve problems lots of ways. Besides our group had a square acetate and it wasn't similar to the rectangular picture, so we couldn't use the rule. We had to find another way! And I think if you look at our different estimates we did a better job than either the police or the rally organizers!

Ms. Morgan: I think you all found a good way of estimating the crowd, and some of you even found a way to use the theorem we all learned. Now I want you to consider the data in the class chart and write a short paragraph telling me whether you think the newspaper or the demonstration organizers had the better estimate. Be sure to explain the basis for your conclusion.

Ms. Morgan planned instruction that would involve students in tasks that call for problem solving, reasoning, and communication and values different approaches to solving problems. **Teachers plan instruction based upon knowledge of subject matter, students, the community, and curriculum goals. (INTASC Principle 7)**

For the final activity of the day, Ms. Morgan gives each group an irregularly shaped picture of a crowd at a rock concert and directs them to estimate the number in attendance. Each of the students must turn in an individual solution but can discuss strategies and the solution with others from their own group.

Ms. Morgan has discovered that, no matter how carefully she monitors individual work, there often are "group" solutions because students who speak other languages somehow manage to communicate. She now often has students work on solutions in their own groups; this allows them to have help on understanding the problem, and places the responsibility for the work back on the student and the groups. She is using a relatively non-threatening individual assessment that still takes individual needs into consideration. **Teachers create instructional opportunities that are adapted to learner diversity. (INTASC Principle 3)**

Vignette Four: Graphic Calculators

The following vignette illustrates how a beginning teacher discovered the importance of technology as an instructional tool and the significance of the teacher's selection of instructional activities in facilitating student learning during a mathematics lesson. This vignette also describes how one teacher, in an effort to make mathematics more meaningful to his students, connects a lesson to the community's interest in school athletics.

The INTASC Principles that will be primarily illustrated are:

> *Principle 2:* Teachers who teach mathematics at any level understand how children learn and develop, and can provide learning opportunities that support their intellectual, social and personal development.
>
> *Principle 4:* Teachers who teach mathematics at any level understand and use a variety of instructional strategies to encourage students' development of critical thinking, problem solving, and performance skills.
>
> *Principle 9:* Teachers of mathematics are reflective practitioners who continually evaluate the effects of their choices and actions on others and who actively seek out opportunities to grow professionally.
>
> *Principle 10:* Teachers of mathematics foster relationships with school colleagues, parents, and agencies in the larger community to support students' learning and well-being.

Mr. Murphy is a beginning teacher who teaches in a high school in a suburban Midwestern community. His class is composed of 25 Algebra 1 students of varied mathematical abilities. The students have been studying linear equations (equations of straight lines) and their characteristics (e.g., slope, x- and y-intercepts). This lesson is the first in an instructional unit that will extend the concept of function to include quadratic equations (equations of parabolas) and their characteristics (e.g., symmetry, breadth, opens "up" or "down", y-intercept, existence of x-intercepts, vertex). Athletics, especially basketball, is important to this community. Both the men's and women's high school teams were in the state championships last year.

The morning's lesson did not unfold as Mr. Murphy had intended. He asked Ms. Kaye, a colleague from within his department if they could meet after school and discuss what went wrong with his lesson and how he might revise it. When they met he described what had happened in the lesson.

"I had planned to use a graphing method to introduce the parabola, using the quadratic equation, $y = ax^2 + bx + c$. I used the Algebra 1 student text, which is ten years old now and does not reference new technology, when planning the lesson. I told the students to divide themselves into groups. I gave them several quadratic equations and instructed them to identify at least five points that would satisfy each equation. Then, they were to graph each set of points on the coordinate axis and connect them. Finally, they were to determine the characteristics of the graph of a quadratic equation (e.g. symmetry, breadth, opens "up" or "down", existence of intercepts, vertex) and summarize their conclusions."

"Unfortunately, the students spent so much time calculating the individual points to graph each equation, they never had the opportunity to summarize their observations. I also noticed that some students chose not to interact with the members of their group and were not engaged in the lesson."

Mr. Murphy uses classroom observation to evaluate the outcomes of teaching and to reflect on practice.

"I've been thinking about the lesson and I see at least two problems. First, the paper and pencil graphing techniques were time-consuming for the students, which prevented the lesson from being completed. Second, my groups were not organized and I did not incorporate any strategy that would encourage students to assume responsibility for their own learning and be actively engaged in the lesson. What would you suggest?"

Mr. Murphy recognizes the importance of engaging students in active learning.

Ms. Kaye shared a discovery activity using the graphic calculators that she had recently prepared for her students, and described some of the positive and negative outcomes she had experienced with the lesson. She also described some strategies that she had used in the past to organize her students into more productive groups. Finally, she recommended that he might consult the support materials in the mathematics office, particularly resources such as the *Addenda Series* from the National Council of Teachers of Mathematics (NCTM).

He draws upon professional colleagues for new ideas.

Teachers continually evaluate the effects of their choices and actions on others and actively seek out opportunities to grow professionally. (INTASC Principle 9)

That evening, Mr. Murphy reviewed several lessons from the *Addenda Series* (NCTM) and read several vignettes in *The Professional Teaching Standards* (NCTM). After reflecting on these, he revised his plans for the next few days. He decided to have his students use graphic calculators. They had had some experience with the calculators in a prior unit and he remembered that they were immediately engaged in the mathematics of those lessons.

Mr. Murphy recognizes the value of incorporating technology into his instruction.

His first priority was to rewrite the worksheet he had given students to maximize the capabilities of the technology. He recognized that the use of calculators would decrease the time necessary for graphing and place an emphasis on exploring patterns across the equations. It would also facilitate the students' explorations by providing a faster, more accurate method for them to test their conjectures. He revised the task by

Teachers plan instruction based upon knowledge of subject matter, students, the community, and curriculum goals. (INTASC Principle 7)

adding a variety of equations that asked the students to make observations, to summarize and to clearly communicate their conclusions (e.g., "all quadratic equations that have a negative coefficient before the quadratic term open downward.")

Worksheet: Discovering Characteristics of Quadratics

Enter the following sets of equations into your calculator. Observe the differences and similarities among the graphs. Write your observations in your notebook and be prepared to summarize the characteristics of the graphs of quadratic equations.

I.
$y = x^2$
$y = x^2 + 2$
$y = x^2 + 4$
$y = x^2 - 3$
$y = x^2 - 5$

II.
$y = (x + 2)^2$
$y = (x + 4)^2$
$y = (x - 3)^2$
$y = (x - 5)^2$

III.
$y = 2x^2$
$y = 1/2x^2$
$y = -2x^2$

IV.
$y = (x + 3)^2 - 2$
$y = (x + 4)^2 + 5$
$y = (x - 5)^2 - 4$
$y = (x - 2)^2 + 1$

V.
$y = (x - 4)(x + 2)$
$y = (x - 1)(x - 6)$
$y = (x)(x + 5)$
$y = (x - 5)(x + 15)$

VI. **Predict what the graph of the following equations will look like. Write your prediction down in your notebook, enter the equation into your calculator and compare your prediction with the actual graph. Were you accurate? If not, why not? What part of the equation changed your prediction?**
$y = 3x^2 - 4$
$y = -6x^2 - 15$
$y = (2x - 5)2^2$
$y = -1/4x + 3$
$y = 2x(x - 2)$

Next he thought about how to group his students for this lesson. Rather than grouping by ability alone, he decided to try giving each student a specific responsibility. In groups of four, he assigned student roles of *teacher*, *grapher*, *recorder* and *reporter to the class*. He felt doing this would capitalize on individual strengths and would keep the focus of the mathematical discourse around the explorations in the lesson. His goal was to stimulate more active engagement of each student. In addition, he hoped this personal accountability would help the students to recognize the importance of each individual in a collaborative effort.

Mr. Murphy believes students should assume responsibility for their learning and make adjustments to instructions to increase engagement and productive work.

Finally, he decided it was important for the students to see a real world connection for the mathematics of this lesson. He recalled a conversation with one of his students earlier in the week. Her sister was a member of the basketball team at the neighboring state university and she was talking about how the coach often videotaped the players. Mr. Murphy decided to call the girl's sister's coach to discuss the possibility of using her videotapes of players shooting free-throws. He described his plan to have the students analyze the parabolic flight of the basketballs and invited her to visit his class at the end of the week. The coach liked the idea and agreed to present her tapes to the class for analyses.

Mr. Murphy seeks to understand students, their families and their communities and uses this information to relate lessons to students' personal interests and pursue problems that are meaningful to them.

Teachers provide learning opportunities that support students' intellectual, social and personal development. (INTASC Principle 2)

As Mr. Murphy reviewed his adjusted plans, he realized that this activity could take as many as two to three days to complete, especially considering the time it would take to explain the new grouping arrangement. He also knew from experience that it sometimes took longer that he expected to explain how to use new function keys on the calculator. However, he felt that the responsibility of group collaboration and the mathematical explorations that the students would experience were worth the time.

The following morning, Mr. Murphy began class by describing his frustration regarding the previous day's lesson. He described the revised schedule for the week, including his invitation to the coach and their plans to share the team videotapes. He challenged the class to learn enough about quadratics in the next few days to describe the mathematics of the flight of a basketball and to use that information to improve their own shots.

Mr. Murphy assigned students to groups and roles, and distributed the worksheets and graphic calculators. He then began to circulate around the room.

While working, Kyle, who was a *grapher* in the group, raised a question.

Mr. Murphy helps students to assume responsibility for using their learning resources and encourages discussion and group interaction. He varies his role as an instructor and facilitator to meet the needs of his students.

Kyle: Are these all supposed to be curves?

Mr. Murphy: Well, who in this group has been assigned the role of *teacher*? Kim, can you help your group out?

Kyle: I'm not sure Mr. M, I think I know but, I'm not totally positive.

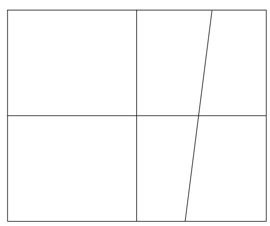
Mr. Murphy: Why don't you talk it over, then change a few coefficients and have your *grapher* key in those equations. Then, see if you can answer your own question.

Mr. Murphy was pleased with the way in which the group was interacting. He made a mental note that the idea of inserting a linear equation on this worksheet caused students to recall their prior knowledge and compare it to the new learning. Although it was difficult for him, he deliberately left the question unanswered to reinforce the role of each member in the collaborative process. He moved on to assess the progress of the other groups.

Mr. Murphy uses classroom observations to evaluate learning. **Teachers continually evaluate the effects of their choices ad actions on others and actively seek out opportunities to grow professionally. (INTASC Principle 9)**

As he monitored the groups, he noticed one group struggling with the graph of an equation that they thought should be a parabola, but which appeared to be linear on the calculator display screen.

George: Mr. M.! Look at my screen! I keyed in y = (x-5) (x+15). I know it is a quadratic and should be a parabola, but it looks like a line. Something is wrong with my calculator!



Mr. Murphy: Wait a minute George, let's figure this out. How do you know it is a quadratic?

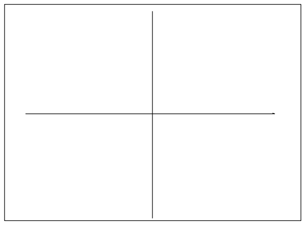
George: Because if you multiply the binomials you get y = x^2+10x-75 and the x^2 makes it quadratic.

Mr. Murphy: Well, did you key it in correctly?

George: Yes, I think so. Mr. M, you try it and Marcy and I will graph it using pencil and paper. Then we will compare them. Come on Marcy.

As Mr. Murphy observed the group working on the problem, a *grapher* from a different group came to him with another difficulty.

Allison:
Mr. Murphy, we just graphed $y=-6x^2-15$ and nothing shows up on the screen.



Mr. Murphy recognized that the difficulty with both groups was due to the fact that their viewing screens did not contain enough of the graphs of the quadratics. The standard viewing screen of the graphic calculator is Domain: {-10 # x # 10} and Range: {-10 # y # 10}. He realized that some of the quadratic equations he assigned had graphs that did not appear in this window and he knew that the groups would need to adjust their calculator screens using the RANGE or WINDOW keys. He also realized that this problem would be experienced by all of the groups and recognized this as an opportunity to extend a prior lesson on domain and range. So he stopped the groups and reconvened the class as a whole.

Mr. Murphy monitors and adjusts his instruction strategies based on classroom observations.
Teachers are reflective practioners who continually evaluate the effects of thier choices and actions on others and who actively seek out opportunities to grow professionally. (INTASC Principle 9)

Mr. Murphy spent the next few minutes questioning the students about the concepts of domain and range that they had studied earlier in the year. He led a discussion of the importance both of the domain and the range to graphing quadratics and for that matter, to any function. He suggested that the groups could use trial and error with the different ZOOM keys to locate the graph or they could use a method of entering x values into the equation, using the Table function, to determine specific points on the graph. This second method would give them more accurate information to set the WINDOW to view the complete graph. After the groups had an opportunity to try each method, he demonstrated how to adjust the viewing window of the calculator screen. Using the method of their choice, they could then test their ideas about what would help them graph the equations that either did not appear on the screen or did not show the complete graph.

Mr. Murphy stimulates student reflection on prior knowledge and helps students link new ideas to prior learning.
Teachers understand how children learn and develop, and can provide learning opportunities that support their intellectual, social and personal development. (INTASC Principle 2)

The students returned to their groups to continue working on their worksheet and Mr. Murphy resumed circulating, to ensure that each group was collaborating. Near the end of the period, he reconvened the class.

He explained the assignment for the next day which was twofold. Since the groups were only able to partially complete the graphing portions of the worksheet, the first part of the homework was for each student independently to make conjectures about the behavior of the graphs of quadratic equations. In their writing, they were to consider the characteristics of the parabolas, their symmetry, x and y intercepts and their vertices. He told the students that he wanted to assess each individual's understanding by giving them the opportunity to process their conclusions on their own initially. The next day they would test these conjectures in their groups with the remainder of the equations.

Mr. Murphy accesses students' thinking and experiences as a basis for instructional activities.
Teachers understand how children learn and develop, and can provide learning opportunities that support their intellectual, social and personal development. (INTASC Principle 2)

The second part of their assignment was to think about a basketball player shooting free-throws. Each student was responsible for drawing a diagram showing the flight of the ball, keeping in mind the real life constraints of the situation (e.g., the height of the gymnasium, the dimensions of a court). Mr. Murphy told the class he anticipated a variety of diagrams from this assignment. There was no "correct" answer. Some students had a better understanding of this exercise because of their personal connection to the sport. However, he wanted to give each individual the opportunity to consider the situation before they began to explore the mathematical analysis the next day.

That evening, reflecting on the day's class, Mr. Murphy was encouraged by the active involvement of every member of the class. The students' use of the graphic calculators gave him an insight into how technology could improve inefficient and outdated instructional techniques. He looked forward to the students testing their hypotheses as well as sharing their diagrams during the next two days. He was particularly excited about the idea of the students working with the coach. He and the coach made plans to freeze-frame the players' free-throw shots so that the students could analyze and mathematically describe the paths of the basketballs. Mr. Murphy made a decision to continue to connect the remainder of the instructional unit on quadratics to the basketball example (e.g., the vertex of the parabola is the maximum height of the ball, a separate equation represents the ceiling; the y-intercept represents the ball leaving the player's hand or the basket).

Mr. Murphy develops links with the greater community and can identify and use community resources to foster student learning.
Teachers foster relationships with school colleagues, parents, and agencies in the larger community to support students' learning and well being. (INTASC Principle 10)

Mr. Murphy noted that the three changes he made to his lesson, incorporating technology, changing the groups and emphasizing real world applications, had significant impact on student

At the end of each lesson, Mr. Murphy reflects on what worked, and what didn't. He uses this information when planning future instruction.
Teachers are reflective practioners who continually evaluate the effects of their

learning. He made a promise to himself to consider each of these aspects whenever he planned future lessons.

choices and actions on others and who actively seek out opportunities to grow professionally. (INTASC Principle 9)

A View to the Future

The articulation of initial licensing standards for mathematics teaching will not, in and of itself, create the kinds of changes we envision in mathematics classrooms. The standards represent a vision for mathematics teaching that can be realized only through the collective efforts of the colleges and universities that prepare teachers; the state and national agencies that approve and accredit teacher preparation programs; state licensure programs that grant initial licenses to teachers; and the professional organizations, colleges and universities, and private companies that provide professional development designed to strengthen mathematics teaching.

Teachers well prepared for the mathematics classrooms of the 21st century will need preparation that is very different from what is offered at most institutions today. The INTASC standards provide a basis for rethinking both the content and structure of teacher preparation programs. Colleges and universities can begin to reexamine the processes and the outcomes of their preparation program by focusing on what teachers know and how they translate what they know into effective practice in a classroom setting. The standards also provide a basis for strengthening the collaboration among faculty in mathematics departments and faculty in colleges of education as they work to provide potential teachers with the knowledge, skills, and dispositions portrayed in the vignettes presented in this document.

INTASC standards provide a framework for consistency between state program review and approval, and voluntary national accreditation; for consistency between state licensure programs and local school hiring decisions. With public consensus on standards for mathematics teaching in place, it will no longer be necessary to base judgments about a potential teacher's effectiveness on "course counting," transcripts, interviews, and tests of basic skills. As performance assessments are developed to mirror teaching standards, more direct evidence of teaching competence can be brought to bear on licensing and hiring decisions. INTASC standards also provide a framework for the design of such performance-based assessments, which can better capture the complexity of teaching.

The development of teachers does not end with the granting of an initial license. Local school evaluation procedures, ongoing state licensure and professional reflection all provide the basis for identifying areas for continued professional development. The INTASC standards can provide coherence here as well. For example, the 10 principles articulated in this document link expectations for beginning teachers to standards for accomplished practice developed by the National Board for Professional Teaching Standards. In this way, the INTASC standards provide coherence between initial and on-going professional development to help teachers excellence at their craft.

Such changes in the system of professional development and licensing of teachers will not occur overnight. However, we believe that the vision presented by the INTASC principles provides a beacon that can guide us in developing teachers who will better prepare our students for the challenges that lie before them.

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